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ECONOMIC ANALYSIS OF PUBLIC OWNERSHIP

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The text, based on the 1990 Germán Bernácer Lectures at the University of Alacant, addresses the question of how should goods and resources be allocated in an economy where the technology is publicly owned and the labor input is privately owned. A solution is advocated where a) if the output is a public good, then users' contributions agree with Lindahl's «benefit principle»; b) if the output is a private good, then a person's consumption of output is proportional to her input contribution, or, equivalently, users end up paying average cost.

1. Introduction

The dominant forms of economic analysis, as exemplified by the Arrow-Debreu theory of markets, simply postulate that all goods and resources are privately owned. Yet this assumption is totally justified neither on positive nor on normative grounds. A large fraction of the initial resources of modern economies, including natural resources, accumulated technical knowledge, and capital invested by the public sector, are owned by society at large, as is a fraction of productive facilities. Asymmetries of information and incentive problems may often favor private ownership on efficiency grounds, yet in many cases some decentralized forms of public ownership may prove superior.

* The text is based on my first two Germán Bernácer Lectures delivered at Alacant, May 2-3, 1990. It presents some themes that have been the subject of my research, much of it in collaboration with Andreu Mas-Colell, of Harvard University, or John E. Roemer, my colleague at Davis. It is a pleasant duty to thank the Department of Economics at the University of Alacant for the opportunity to deliver the Lectures and for interesting discussions with members of the audience. Morevorer, I am indebted to Ignacio Ortuño, to John Roemer, to Antonio Villar and to the Editor of *Investigaciones Económicas* for many useful comments and suggestions on the present text, and to Sati Madani for research assistance. Of course, they should not be responsible for any remaining mistakes, shortcomings or idiosyncracies. Finally, I acknowledge the hospitality of the Department of Economics and Economic History at the *Universitat Autònoma de Barcelona* and the Institute of Economic Analysis, CSIC, which provided a stimulating and pleasant environment for the preparation of the Lectures.

The present text maintains the hypothesis that a certain good or resource is publicly owned, whereas other productive goods are privately owned. I will often refer to the public ownership of a technology or a firm, which, when returns to scale are decreasing, can also be interpreted as the public ownership of an implicit input, as in Lionel McKenzie (1959).

How should the benefits derived from a publicly owned technology be allocated? Section 2 below focuses on the production of public goods, and develops the Lindahlian approach into a theory of cost sharing. The cost-sharing solution can be rephrased as a solution concept for the distribution of the surplus of a publicly owned firm that supplies public goods.

The solution concept is in turn applied to the private good case in Section 3, which also discusses alternative approaches and applies them to three alternative institutional scenarios, namely a public firm, a cooperative and a common pool resource.

2. Sharing the cost of a public good

2.1. Public and private goods

Economic analysis is largely concerned with private goods: goods that are «rival in consumption» in the sense that their consumption by one person decreases their availability to other individuals. For example, the bigger the piece of apple consumed by Eve, the smaller will be the portion that Adam may enjoy. Together with production and exchange, consumption is often considered a basic economic activity. Economists frequently refer to the members of a society as «consumers», not «individuals» or «citizens», or, for that matter, «users». We say «consumption» and not «enjoyment» or «use». If we consult a dictionary we find out that the word «consume» is equated to «destroy» or «slowly destroy». This is due to the fact that the enjoyment or use of a private good always goes hand in hand with the total or partial destruction of the good.

But many goods can be used without being destroyed and, therefore, are not necessarily private. For example, a nature loving hiker, one who «leaves nothing but footprints and takes nothing but photographs», enjoys the mountain and the forest without destroying them, whithout diminishing others' capacity to enjoy them as well. The forest can therefore be considered a public good. But the same forest can be used as a private good, say, if cut and converted into firewood. It would often be preferable to refer to the public (non-destructive) or private (destructive) use of a good, rather than to public or private goods.

Some goods can only be used in specific ways because of physical constraints or basic reasons of efficiency. For instance, gasoline is useful only if burned, and for this reason it is intrinsically a private good. Other goods, such as information, can and should be used without being destroyed, and therefore are public goods. Last, there are intermediate examples of goods that can be used up to a certain level, determined by capacity limits, without being destroyed. Some natural environments fall into this category.

We can characterize the two extremes of purely private goods and purely public ones in the following manner. The total available quantity of a private good is the sum of the quantities that different individuals can enjoy. On the other hand, the total available quantity of a public good equals the quantity each individual can enjoy. Note that I say «can enjoy» and not «actually enjoys». For some public goods, the two concepts coincide. For others, the difference is related to two characteristics: *free disposal* and the possibility of *exclusion.*

A public good displays *free disposal* if an individual can decide to enjoy an amount lower than the available quantity, and it is *excludable* if the supplier can decide who has access to it. Television is an example of a public good that displays free disposal. If a TV station offers 20 hours of programming per day, each user can enjoy, if she so desires, all 20 hours. But she may decide to watch TV for only two hours. We define the quantity of the public good TV to be 20 hours, no matter how many hours each person watches TV.

National defense, on the other hand, does not display free disposal. If the armed forces have 300 fighter planes, then no private citizen can decide to benefit from only 200 planes. The available quantity, namely 300, coincides here with the actual use.

The TV example also illustrates the concept of exclusion. Regular television is transmitted through waves accessible to everyone: nobody can be excluded from viewing it. Cable TV (or the scrambling of the broadcast signal), on the contrary, allows for exclusion. In general, free disposal can exist with of without exclusion, while it is difficult to find examples of goods that are excludable but do not display free disposal.

2.2. *Private and public ownership*

Economic analysis often postulates private property rights for all the goods and resources in the economy. Of course, the distinction between private and public ownership does not coincide with the private or public character of the good owned. For instance, many TV facilities (which produce a public good) are privately owned, and many oilfields (private goods) are publicly owned.

As just mentioned, a particular good, say a deer in a public forest, can be alternatively used as a public good (say, when it is being photographed) or as a private good (when it is turned into venison sausages). It is, in principle, publicly owned but, in the second case, it is «privatized», so to speak, by the hunter who acts within the limits of the law. The laws can allow for different levels of privatization. For example, a hunter from Alacant has the right to sell his quarry, while in the United States the level of privatization is lower: the hunter has the right to eat or stuff her quarry, but she cannot sell it. In fact, in the United States game can typically be consumed only in private homes and not in restaurants. The private ownership of goods and resources usually assumed in economic analysis, on the contrary, includes the right to use as well as abuse, destroy or sell.

2.3. Individual decisions and collective decisions

Economic analysis often postulates individualistic private ownership. But there is an exception, namely, the ownership of corporations, shared among many people in accordance with the distribution of its equity. This form of property is somewhat restricted because, whereas it grants the shareholder voting rights and the entitlement to a dividend, it does not allow for participation in the day-to-day decisions of the firm. One may expect the firm's management to act in the interests of the shareholders. For instance, it is often understood that all shareholders are interested in profit maximization: management decisions to ensure profit maximization are in fact implementing a (perhaps implicit) decision by the shareholders. Let us accept this argument for the time being even though, as I will discuss in Section 3, it implicitly assumes that the shareholders do not work in the firm or consume its output.

The quantity of private goods bought, sold, produced or consumed is determined, in most economic models, by the interaction within the market of decisions based on initial property rights. These are either personal decisions to sell some goods and to use personal wealth to acquire some other goods, or decisions that firms make in the owners' interests. These decisions pursue individual objectives.

Such modeling is inadequate when dealing with goods, resources or enterprises that are public. A public enterprise belongs to all citizens, including its workers and the consumers of its products: its decisions are collective ones, and pursue social objetives. A public good that displays free disposal and excludability (say, cable TV) may be supplied by private firms. But its market is typically thin, because technological efficiency requires a limited number of suppliers. Laissez-faire would then typically lead to monopolistic or oligopolistic inefficiencies, and some form of regulation is usually required in order to protect the public interest. The decisions of public firms or regulatory commissions are public decisions.

2.4. Lindahl equilibrium: unanimity and the benefit principle

Here we consider decisions on public goods, whereas Section 3 below will refer to the decisions of public firms that produce private goods. They are both social decisions, and they have been studied not only by economists but also by political scientists, social philosophers and sociologists.

The main contribution of economics to the analysis of decisions involving public goods is due to the Swedish economist Erik Lindahl (1919). He was inspired by the ideas of Knut Wicksell, his adviser, who in turn was influenced by the benefit approach of the Italian finance school. Three ideas are fundamental in this approach. The first one is the «principle of the specialization of the budget», namely the notion that one should simultaneously determine the quantity of the public good to be supplied and each person's contribution towards covering its cost. The second one is the «benefit principle», i.e., such contributions should take into account the benefit each person derives from the public good or service. Last, the argument that, because the quantity of the public good is the same for everyone, any decision to supply a public good should, in some form, be unanimous. The unanimity was anticipated by authors preceding Lindahl, but he presented it with greater clarity. For example, Knut Wicksell wrote in 1896:

> «As long as the project in question permits the creation of utility beyond its cost, it would always be theoretically possible, and often feasible in practice, to find a cost-sharing scheme such that all parties consider the project beneficial, so that their unanimous approval would be possible.»[1]

Lindahl considers the decision of a two-party parliament, one party representing the poor classes, the other representing the well-to-do classes. There is a twofold decision to be made: how much of a certain public good to be produced, and how to distribute its cost between the classes. Lindahl's originality consists in presenting the two problems as two aspects of a single problem, namely to find a cost-sharing scheme that induces the parties to choose the same quantity of the public good. If costs are shared equally, then the poor may prefer a lesser amount than the rich. Assuming that, if there is disagreement, then the actual level of production is the lower of the two quantities, it would be in the interest of the rich to subscribe a greater porportion of the cost and ensure that the poor accept a higher level of the public good. A Lindahl equilibrium consists of a share of the cost to be paid by the rich and a share of the cost to be paid by the poor (adding up to one hundred percent) such that the quantities of the public good that rich and poor choose, given their cost shares, coincide.

Figure 1, taken from Erik Lindahl (1919), illustrates the idea. There is a public good and a private good. The quantity of public good is denoted z. Assume that there are M_1 rich persons, and that the utility that a rich person obtains from z units of the public good and x units of the private good is $x + w_1(z)$: preferences are quasilinear, and w_1 is the «benefit function» (or «willingness-to-pay function»). If the cost of producing z units of the public good is kz, in units of the private good, and if s_1 is the fraction of the cost paid by the rich, then the party representing them will attempt to obtain a level of z that maximizes $M_1 w_1(z) - s_1 kz$, i.e., (assuming differentiability and interiority), a level of z satisfying $M_1 w_1'(z) = s_1 k$, where w_1' is the derivative of w_1. The solid curve in Figure 1 represents the relationship between the quantity of the public good chosen by the rich and s_1. The

[1] Wicksell (1896), translated into English by Richard Musgrave and Alan Peacock (1958) and quoted by John Roberts (1974).

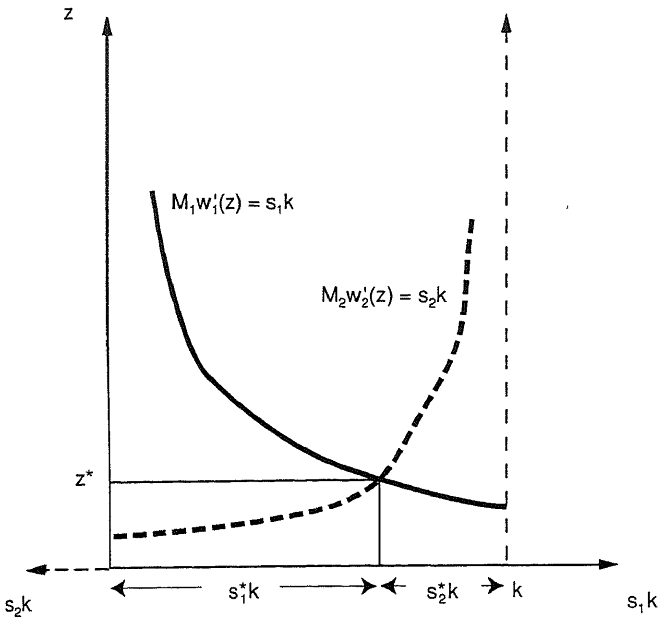

Figure 1
Lindahl's graph.

equation $M_2w_2'(z) = s_2k$ is the relevant one for the poor class. Because, by definition, $s_2 = 1 - s_1$, this equation can be represented by the dashed line in Figure 1, which reverses the direction of the horizontal axis. The point indicated by Lindahl is denoted s_1^*k and z^*. The rich choose the quantity z^* given the fact that they have to pay the fraction s_1^* of the cost. The poor choose the same quantity, given that they must pay the fraction $s_2^* = 1 - s_1^*$ of the cost. The decision is unanimous.

On the other hand, the solution agrees with the benefit principle because the contribution of each person (or class) is proportional to the marginal benefit provided by the public good. Indeed, the (aggregate) contribution of the rich class is $s_1^*kz^* = M_1w_1'(z^*)z^*$, i.e., the quantity of public good multiplied by its marginal utility, whereas the poor contribute $s_2^*kz^* = M_2w_2'(z^*)z^*$. Thus:

$$\frac{\text{Contribution of a rich person}}{\text{Contribution of a poor person}} = \frac{\dfrac{s_1^*kz^*}{M_1}}{\dfrac{s_2^*kz^*}{M_2}}$$

$$= \frac{w_1'(z^*)z^*}{w_2'(z^*)z^*} = \frac{\text{Marginal valuation by a rich person}}{\text{Marginal valuation by a poor person,}}$$

in accordance with the benefit principle.

I have assumed that costs are proportional to the quantity of public good, i.e., the per-unit cost of providing the public good is a constant k. Thus, we can say either that the rich pay the fraction s_1 of the cost kz, or that the rich pay a price s_1k per unit of the public good. This assumption, implicit in Lindahl, that costs are proportional to quantity is nothing but the hypothesis of constant returns to scale.

2.5. *The common version of the Lindahl model*

Modern textbooks present the common version of Erik Lindahl's ideas, a version developed in the late 1960's under the influence of the Arrow-Debreu model of competitive general equilibrium. The most influencial article was that of Duncan Foley (1970), but Françoise Fabre's simultaneous and independent work (1969, 1970), the later elaborations of Jean-Claude Milleron (1972) and John Roberts (1974) as well as Leif Johansen's preliminary formulations (1963) should be mentioned. Foley (1970) keeps the constant-returns-to-scale hypothesis, but others (like Fabre, 1969, 1970 and Milleron, 1972) allow for decreasing returns to scale. The common version has the generality of the Arrow-Debreu model in the sense that it allows for different individuals and many goods, both public as well as private. Moreover, it exploits the parallelism with the Arrow-Debreu model to show that the resulting equilibrium is Pareto efficient: more on this in Section 2.11 below.

The common version generalizes Erik Lindahl's original model at the cost of obscuring its interpretation. The number of public goods is generalized from one to N, $N \geqslant 1$. The parliament and the negotiations between parties disappear. Instead, one refers to prices and profit maximization, suggesting the operation of a competitive market, and views the law of demand and supply in imaginary markets as determining the prices and quantities of the public goods. There are M persons. Person i pays the «personalized» price p_{ij} for a unit of public good j ($j = 1,\ldots, N$), i.e., she pays $p_{ij}z_j$ if the quantity supplied is z_j, whereas the firm that supplies public good j receives the sum of the p_{ij}'s, that is $\Sigma_{i=1}^{M} p_{ij}$, for each unit of good j supplied. The firm treats $\Sigma_{i=1}^{M} p_{ij}$ parametrically and chooses the quantity of public good in accordance with the profit maximization criterion. Of course, profits are zero under constant returns to scale.

What real-life institution is captured by the common version of the Lindahl model? Some public goods are directly supplied to consumers by private corporations. Consider again a private cable TV corporation, which receives as revenue the sum of the amounts paid by the consumers. We can recognize Foley's pseudo-market in this example, even though the connection is typically weak to the extent that a small number of firms supply the market, and they should not be expected to treat prices parametrically.

But it is often the public sector that simultanously contracts with the supplier the quantity to be produced and the total payment, a desirable procedure when exclusion is impossible, costly or inefficient. Foley's pseudo-market lacks a real-life counterpart in these cases.

2.6. *Who produces the public good? Variable returns to scale and the distribution of profits and losses*

Foley (1970) assumes profit maximization and constant returns to scale. It hardly matters, under these assumptions, who owns the corporations that produce the public goods, because no benefits or losses are to be distributed among shareholders. Moreover, the «personalized price» p_{ij} can be interpreted as the share s_i that appears in Lindahl, i.e., as the fraction of the unit cost of good j that citizen i has to cover.

The property of constant returns to scale implies the two conditions of *additivity* and *divisibility* of the productive processes (see Andreu Mas-Colell, 1987). As mentioned above, the common version of Lindahl's model allows for variable returns to scale. But profit maximization is impossible when returns to scale are increasing, i.e., when productive processes are additive but not divisible. Hence, no equilibrium exists, or, in other words, the model is inappropriate for increasing returns.

Decreasing returns to scale (productive processes are divisible but not additive) present no problems for the existence of equilibrium. But they worsen the difficulties in interpreting the pseudo-market because, in addition to its previous tasks, it now generates profits that must be distributed among shareholders. Mimicking the function of competitive markets, the pseudo-market now determines not only the quantities of public goods and the allocation of their costs, but also profit incomes. The model predicts, in particular, that the revenue of the firm producing good j will equal $z_j \sum_{i=1}^{M} p_{ij}$, an amount larger than the production costs. The formulation is again inappropriate for the case where a public agency simultaneously contracts both the quantity and the total payment, because the agency will often be able to exert monopsonist power and keep payments at the level of the opportunity cost of the supplier.

2.7. *The implicit input in competitive markets*

An influential idea in economics is that profits are nothing but the payment for an implicit production input, valued at its marginal product by competitive markets. Lionel McKenzie (1959) formulated the idea clearly, but it was present in one form or another much earlier, and is rooted in the Ricardian theory of economic rent.

As noted in Mas-Colell (1987), if a firm generates positive profits at a competitive equilibrium, then its technology necessarily violates the additivity assumption: in other words, if we start from an input combination that allows us to produce the equilibrium quantity, say Q units of output, and double the quantities of all inputs, then the quantity of output that can be obtained will be *less* than $2Q$.

But how can this happen? The absence of additivity requires, almost by logical necessity, the existence of something that affects production but is not included in the input list, because an exact clone of the necessary elements to produce Q units should be able to produce another Q units.

By definition, the input list includes all the goods and services that affect production and are transacted in a market. Therefore, if a competitive firm has positive profits, then there is something, let us call it the «implicit input», that affects production but that the firm cannot purchase in the market.

McKenzie developed the following idea: consider a competitive firm that produces good z using labor, denoted x, in accordance with the production function $z = f(x)$. Assume that if the price of output is p^* and the wage is w^*, then the firm maximizes profits by buying x^* units of labor and selling $f(x^*)$ units of output, thus obtaining a profit of $\pi^* = p^* f(x^*) - w^* x^*$. This profit can be viewed as the price of the implicit input as follows.

Consider a modified economy where our firm uses labor and a new input, called the implicit input, denoted ξ. The modified production function is: $F(x,\xi) = \xi f(x/\xi)$, if $\xi > 0$, and $F(x,\xi) = 0$, if $\xi = 0$. It is clear $F(x,\xi)$ displays constant returns to scale. The function F can be graphically constructed as follows. Figure 2 depicts the original production function f in two dimensions whereas Figure 3 displays a three-coordinate system, with axes denoted x, ξ and z. Place the graph in Figure 2 on the coordinate system of Figure 3 in such a manner that it is parallel to the (x,z) plane and intersects the ξ axis at $\xi = 1$. Now join each point in the transported graph of the f function with a half line that starts at zero and extends infinitely to the right. This yields the graph of the function F, a generalized cone.

I show now that, if the price of output and the wage are p^* and w^*, as before, and the price of the implicit input is π^*, then the firm maximizes profits by buying x^* units of labor and one unit of the implicit input, thus producing the quantity $F(x^*,1)$, which is equal to $f(x^*)$. The profit function of the firm is now $\Pi(x,\xi) = p^* F(x,\xi) - w^* x - \pi^* \xi$, satisfying the condition $\Pi(x^*,1) = 0$. If $(x^*,1)$ did not maximize $\Pi(x,\xi)$, then there would exist a combination of inputs (x',ξ') such that $\Pi(x',\xi') > \Pi(x^*,1) = 0$, i.e., such that $\xi' > 0$ and $p^*\xi' f(x'/\xi') - w^* x' - \pi^* \xi' > 0$, or, dividing by ξ', such that $p^* f(x'/\xi') - w^* x'/\xi' > \pi^*$. But this would mean that, in the original economy, the firm could obtain higher profits by using x'/ξ' units of labor than by using x^*, contradicting the assumption that the firm maximized profits at x^* in the original economy.

If the function f is differentiable, then the same idea can be conveyed by saying that the competitive profits of the original economy are the marginal value product of the implicit input in the modified economy at the point where the firm uses one unit of the implicit input. Indeed, the partial derivative of $F(x,\xi)$ with respect to ξ is $\partial F/\partial \xi = f(x/\xi) + \xi f'(x/\xi)\,(-x/\xi^2)$,

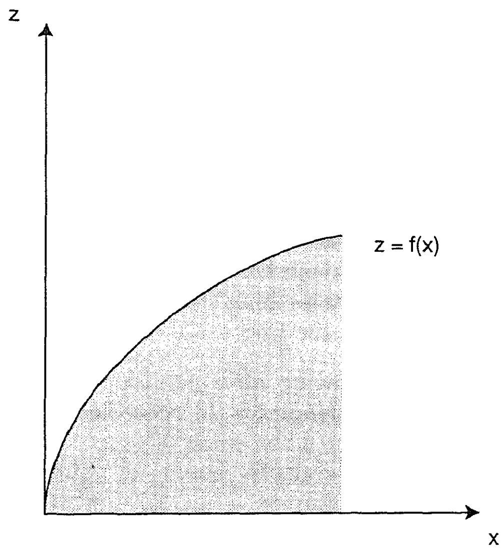

Figure 2
A production function displaying. Decreasing returns of scale.

equal to $f(x^*) - x^*f'(x^*)$ when $(x,\xi) = (x^*,1)$. Now the partial derivative of $F(x,\xi)$ with respect to x is $\partial F/\partial x = \xi f'(x/\xi)/\xi$ equal to $f'(x^*)$ at the point $(x^*,1)$. By the perfect competition assumption, $f'(x^*) = w^*/p^*$. Therefore, at the point $(x^*,1)$ we have that:

$$\partial F/\partial\xi = f(x^*) - w^*x^*/p^* = \pi^*/p^*$$

i.e., the profit π^* is the value of the marginal product of the implicit input.

2.8. Private ownership a nd public ownership of the implicit input

The common version of the Lindahl model views the scheme for distributing profits as an exogenous datum. It postulates a private ownership economy: each person initially owns shares in the firms in addition to the initial endowment of goods. The proportion of the shares of firm j that person i owns is represented by θ_{ij}. All the shares are in the hands of the consumers, i.e., $\Sigma_{i=1}^{M} \theta_{ij} = 1$.

We may follow the interpretation in the previous section and view the implicit input as a productive resource or service devoid of direct utility. Only one unit exists, initially owned by shareholders in the proportions described by the parameters θ_{ij}. The profit of firm j received by consumer i can be interpreted as the income derived from selling θ_{ij} units of the

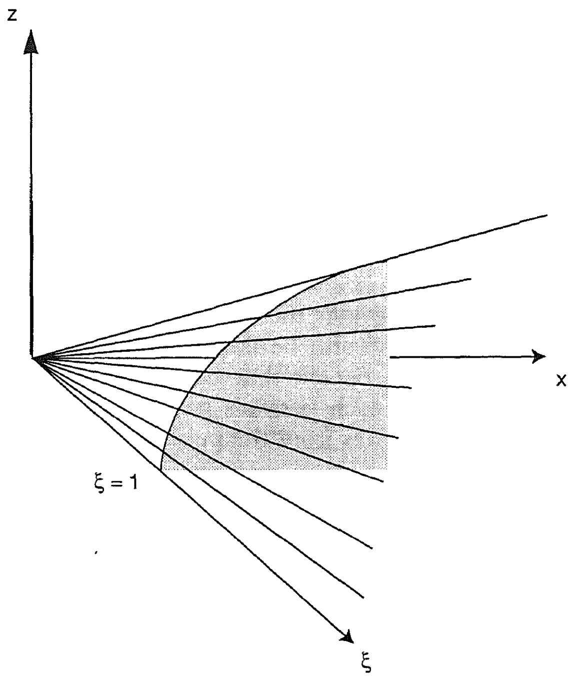

Figure 3
Constant returns with an implicit input.

implicit input to the firm. In fact, Lionel McKenzie (1959) adopts this interpretation of the implicit input as a privately owned good, and uses the expression entrepreneurial factor to describe it. (Andreu Mas-Colell, 1987, called it the «entrepreneurial will.»).

This interpretation suggests that the implicit input is a productive service provided by the people who introduce a new product, launch an enterprise or bear its risk. To the extent that the entrepreneurial capacity is a human resource, either innate or acquired through investment in experience and education, it makes sense to consider it as private property. The proportion of shares that shareholder *i* owns then reflects her endowment in entrepreneurial capacity, a human resource not much different from work capacity, reflected in her initial endowment of labor time.

But the implicit input does not have to be a privately owned good in all its interpretations. Even if it is interpreted as the services of those who innovate, start new enterprises or bear corporate risks, it is precisely the public sector that performs these functions in the supply of many public goods.

The entrepreneurial factor should then be viewed as public property. The implicit input should be interpreted as a natural resource in other cases, as for instance the water in an aquifer, owned in principle by the community and not by private individuals: Section 3 below discusses this in greater detail.

2.9. The benefit principle in taxation

Erik Lindahl (1919) presents his analysis both as an idealized description of what occurs in a parliament and as a just manner of distributing costs. Lindahl writes that, assuming that the initial distribution of private goods is fair, the final allocation will also be fair in the sense that each citizen's contribution agrees with the benefit that she obtains. As seen in Section 2.4, at the Lindahl solution each person's payment is proportional to the marginal benefit that she derives from the public good.

Is the benefit principle upheld in the common version of the Lindahl model? The answer is «yes» when returns to scale are constant, but «no» when they are decreasing. Consider an economy with a single private good, a single, nonexcludable public good, and two people, one rich (subscripted by 1) and one poor. The private good is used in the production of the public good, and the technology is described by a cost function (i.e., the inverse of a production function) $G(z)$. In words, $G(z)$ is the quantity of the private good necessary to produce z units of the public good. Denote its derivative (the marginal cost) by $G'(z)$. We assume that returns to scale are decreasing: more precisely, that $G(z)$ is a convex function ($G''(z) > 0$). The private good is a resource of which the rich initially owns ω_1 units, and the poor ω_2 units.

Consider the common version of the Lindahl model, in which the firm's profits are distributed according to an exogenous rule, say that the rich receives all profits (i.e., $\theta_1 = 1$ and $\theta_2 = 0$). The equilibrium of the common version includes a quantity of public good z^*, a personalized price for the rich $p_1^* = w_1'(z^*)$, a personalized price for the poor $p_2^* = w_2'(z^*)$, and the level of profits $\pi^* = (p_1^* + p_2^*)\, z^* - G(z^*)$.

What is the contribution of the poor to the provision of the public good? If z were zero, then the poor's consumption of the private good would be ω_2 units, while now, the poor enjoys z^* units of the public good, but only $\omega_2 - p_2^* z^*$ units of the private good. Therefore, the poor contributes $p_2^* z^*$ units of the private good to the provision of z^* units of the public good.

We can proceed in the same manner to compute the contribution of the rich. If z were zero, then her consumption of the private good would be ω_1, while now the rich enjoys z^* units of the public good and $\omega_1 - p_1^* z^* + \pi^*$ units of the private good. Therefore, the rich contributes $p_1^* z^* - \pi^*$ units of the private good to the provision of z^* units of the public good. It is clear that, to the extent that $\pi^* \neq 0$, such an allocation does not satisfy the benefit principle, because:

$$\frac{\text{Contribution of the rich}}{\text{Contribution of the poor}} = \frac{p_1^* z^* - \pi^*}{p_2^* z^*} = \frac{w_1'(z^*) z^* - \pi^*}{w_2'(z^*) z^*}$$

$$\neq \frac{w_1'(z^*) z^*}{w_2'(z^*) z^*} = \frac{\text{Marginal valuation of the rich}}{\text{Marginal valuation of the poor}}$$

2.10. *The core of the economy*

Duncan Foley (1970) showed that, under constant returns to scale, the equilibrium is not only Pareto efficient, but it also has the desirable property of belonging to the core of the economy. Recall that an allocation belongs to the core if no group of individuals (or in the language of game theory, no coalition) can improve on it by isolating itself from the rest of the society and using only the resources of its members. This is a property of social stability: an allocation not in the core is subject to a centrifugal tension because some individuals could do better by repudiating the allocation and relying solely on their own resources.

As Thomas Muench (1972) noted, public-good economies typically have large cores, because small groups find it difficult to improve their lot by producing the public goods on their own. Yet under decreasing returns to scale the equilibrium of the common version of the Lindahl model may not belong to the core. The following example illustrates. Let the inverse production function be

$$G(z) = \begin{cases} z/4 & \text{if } z \leq 4, \\ -3 + z & \text{if } z \geq 4. \end{cases}$$

Assume that the public good is useless to the rich, that is: $w_1(z) = 0$ for all z, while the willingness-to-pay function of the poor is $w_2(z) = 6z - z^2/2$. The rich receives all profits, i.e., $\theta_1 = 1$ and $\theta_2 = 0$. It is easy to verify that, in the equilibrium of the common version of the Lindahl model, the personalized prices are $p_1^* = 0$ and $p_2^* = 1$, and the level of public good is $z^* = 5$, which requires the use of $G(5) = 2$ units of the private good in production. (The poor chooses z in order to maximize $w_2(z) - p_2^* z = 5z - z^2/2$ and, thus, she chooses $z = 5$). The poor's contribution is $p_2^* z^* = 5$, more than $G(5)$. If the poor were to leave society *while having access to the same technology*, then she could obtain the five units of public good by only contributing its cost $G(5) = 2$. Therefore, the equilibrium does not belong to the core.

The underlined words, crucial in the argument, suggest that the concept of the core does not fit well with the private ownership of technologies with decreasing returns to scale. As seen in Section 2.7, decreasing returns to scale require the existence of an entity that affects production but is not included in the list of inputs. If this entity is a privately owned factor of production, then it should not be included in the resources of every coalition that threatens to break away. In our example, if the production of the public good requires an entrepreneurial factor that only the rich possesses, then the poor should be considered incapable of autarkically producing any quantity of public good.

2.11. Cost-share equilibrium

The interpretation difficulties discussed above have motivated a series of papers that return to Erik Lindahl's (1919) original idea of «percentages of cost to pay» in place of the «personalized prices» of the common version of the Lindahl model. This line of research includes articles by Mamoru Kaneko (1977), Shlomo Weber and Hans Wiesmeth (1991), as well as my collaboration with Andreu Mas-Colell (1989). On the one hand, the focus is more in line with the benefit principle than the common version of the Lindahl model. On the other hand, it allows for the extension of the analysis to increasing return technologies, which are especially relevant in the production of public goods.

Let us return to Erik Lindahl (1919) and Figure 1. As I mentioned, the parameters s_1 and s_2 are interpreted as cost shares, and if the cost is $G(z)$, then the rich pays $s_1 G(z)$, the poor pays $s_2 G(z)$, and because $s_1 + s_2 = 1$, we have that $s_1 G(z) + s_2 G(z) = G(z)$. Generalizing this idea to M individuals and N public goods, we define a *cost-share system* as a M-tuple of functions $g_i(z)$, $i = 1,..., M$, such that for all z, $\Sigma_{i=1}^{M} g_i(z) = G(z)$, where $z = (z_1,..., z_N)$ is an N-dimensional vector. Person i initially has ω_i units of the private good, and her preferences are represented by the utility function $U_i(x_i, z)$, where x_i is her final consumption of the private good. Note that this formulation relaxes the quasilinearity assumption «$U_i(x_i, z) = x_i + w_i(z)$». We define a *cost-share equilibrium* as a vector of public goods z^* and a cost-sharing system $\{g_i(z) \mid i = 1,..., M\}$ such that for each person i, z^* maximizes:

$$U_i(\omega_i - g_i(z), z). \quad [1]$$

It is important to observe that a cost-share equilibrium is always efficient. Moreover, if $g_i(z) \geq 0$ for all i and z (i.e., the functions $g_i(z)$ are always non-negative, or, in other words, the individuals never «contribute» negative quantities), then it is easy to show that a cost-share equilibrium belongs to the core of the economy[2]. In other words, a sufficient condition for an allocation to belong to the core is that it is a cost-share equilibrium for non-negative functions $g_i(z)$.

It turns out that, under some assumptions, being a cost share equilibrium is also a necessary condition for an allocation to belong to the core, although the proof is much more complex. Weber and Wiesmeth (1991) show that, if there is only one public good and the function $G(z)$ is convex and differentiable (nonincreasing returns to scale), then any allocation in the core can be supported as a cost-share equilibrium.

2.12. *Linear cost-share equilibrium*

So far no conditions have been imposed on the functions $g_i(z)$. A useful restriction is to limit attention to functions that are linear in the quantities of public goods and in their production cost, i.e., functions of the form:

$$g_i(z) = \Sigma_{j=1}^{N} a_{ij} z_j + b_i G(z), \qquad i = 1,\ldots, M,$$

with parameters satisfying the conditions:

$$\Sigma_{i=1}^{M} b_i = 1, \qquad [2]$$

$$\Sigma_{i=1}^{M} a_{ij} = 0, \quad j = 1,\ldots, N. \qquad [3]$$

The two conditions can be justified as follows. By assumption, for all z, one must have $\Sigma_{i=1}^{M} g_i(z) = G(z)$, i.e.,

$$\Sigma_{j=1}^{N} z_j(\Sigma_{i=1}^{M} a_{ij}) + G(z)\, \Sigma_{i=1}^{M} b_i = G(z).$$

Hence, it is natural to impose the condition $\Sigma_{i=1}^{M} b_i = 1$, which in turn requires that $\Sigma_{j=1}^{N} z_j (\Sigma_{i=1}^{M} a_{ij}) = G(z) - G(z) = 0$, for all z, i.e., $\Sigma_{i=1}^{M} a_{ij} = 0$.

If the functions $g_i(z)$ of the cost-share system are of this type, then the corresponding equilibrium is called a *linear cost-share equilibrium.*

Let a technology, described by the inverse production function $G(z)$, and M persons, described by their utility functions U_i and their initial endowments of private good ω_i, be given. Can one find parameters a_{ij} and b_i, and quantities z_j of public goods ($i = 1,\ldots, M, j = 1,\ldots, N$) that constitute

[2] *Proof.* Suppose not, i.e., let there be a subset S of $\{1,\ldots, M\}$ and an allocation for such a subset $\{(z', x_i') \mid i \in S\}$ such that, for all $i \in S$, $U_i(z', x_i') > U_i(z^*, x_i^*)$, where (z^*, x_i^*) are the quantities corresponding to the cost-share equilibrium. The fact that $U_i(z', x_i') > U_i(z^*, x_i^*)$ implies that $x_i' > \omega_i - g_i(z')$, since otherwise (z^*, x_i^*) would not mazimize $U_i(z, \omega_i - g_i(z))$. Thus, $\Sigma_{i\in S} x_i' > \Sigma_{i\in S} \omega_i - \Sigma_{i\in S} g_i(z')$. But, by the non-negativity of the $g_i(z)$ functions, we have that $\Sigma_{i\in S} g_i(z') \leq \Sigma_{i\in\{1,\ldots,N\}} g_i(z') = G(z')$. Hence, $\Sigma_{i\in S} x_i' > \Sigma_{i\in S} \omega_i - G(z')$, contradicting the hypothesis that the coalition S can reach the prime allocation by using its own resources. QED. The same argument proves efficiency, now without requiring the nonnegativity condition.

a linear cost-share equilibrium? Let us count the equilibrium equations and unknowns for the differentiable, interior case. We have NM parameters a_{ij}, M parameters b_i and N quantities z_j, i.e., $NM + M + N$ unknowns. Person i maximizes [1], which, in a linear cost-share system, becomes:

$$U_i(\omega_i - \Sigma_{j=1}^{N} a_{ij} z_j - b_i G(z), z).$$

Thus, we have, for each person, the N first order conditions for utility maximization:

$$(\partial U_i/\partial x_i)\,[-a_{ij} - b_i\, \partial G\,(z)/\partial z_j] + \partial U_i/\partial z_j = 0,\ \ j = 1,..., N,$$

i.e., MN equations in total for M persons, to which we must add condition [2] and the N conditions [3]. Altogether, we have $MN + M + N$ unknowns, but only $MN + 1 + N$ equations: the accounting suggests the presence of $M-1$ degrees of freedom.

My article with Andreu Mas-Colell (1989) shows that, indeed, if all functions are differentiable and $G(z)$ is convex, then the model does have $M-1$ degrees of freedom, which moreover can be easily interpreted in terms of the common version of the Lindahl model. We show that to each linear cost-share equilibrium corresponds an equilibrium of the common version and vice-versa. The correspondence between the different parameters is as follows. In the cost-share equilibrium we have the parameters a_{ij} and b_i, while in the common version of Lindahl model we have the personalized prices p_{ij} $(i = 1,..., M, j = 1,..., N)$ and the exogenous profit shares θ_i $(i = 1,..., N)$. The relationship between the two sets of parameters at the respective equilibria is given by:

$$b_i = \theta_i, \qquad i = 1,..., M. \qquad [4]$$

$$p_{ij} = a_{ij} + b_i\, \Sigma_{h=1}^{M} P_{hj},\ i = 1,..., M,\ j = 1,..., N. \qquad [5]$$

Thus, the concept of linear cost-share equilibrium may capture any exogenous vector of profit shares of the firm that produces the public goods, or according to the interpretation of Section 2.8, any initial distribution of the property rights over the input implicit in its technology.

2.13. Balanced, linear cost-share equilibrium

As I just pointed out, the concept of linear cost-share equilibrium has $M-1$ degrees of freedom, which correspond to the profit-share parameters, and thus it can reflect any initial distribution of private property rights on the implicit input. Indeed, by exogenously fixing the b_i parameters we can represent an initial distribution where person i owns the fraction b_i of the implicit input. But, by the same token, the concept is undetermined when the technology is publicly owned.

We can close the model by adding $M-1$ equations of the following form. In a linear cost-share equilibrium, the contribution of one person, say i, is the sum of the two terms $\Sigma_{j=1}^{N} a_{ij} z_j^*$ and $b_i G(z^*)$. Given that $\Sigma_{i=1}^{M} b_i = 1$, the sum over all individuals of the second terms, $b_i G(z^*)$, equals $G(z^*)$, while the sum of the first terms is zero. In other words, the second terms cover the costs of producing z^*, while the first terms, which can be positive for some individuals and negative for others, are in the nature of interpersonal transfers.

We may require, as a condition for redistributive neutrality, that such transfers be zero, i.e., that the first term of each person's contribution satisfy, at equilibrium, the *no-transfer condition:*

$$\Sigma_{j=1}^{N} a_{ij} z_j^* = 0, \qquad i = 1,..., M. \tag{6}$$

This adds $M-1$ independent equations to the equilibrium conditions, nullifying the previous $M-1$ degrees of freedom. (There are $M-1$ and not M because, as I just mentioned, the sum of the first terms is zero, i.e., $\Sigma_{i=1}^{M} \Sigma_{j=1}^{N} a_{ij} z_j^* = 0$.) An equilibrium that satisfies the $M-1$ additional conditions [6] is called a *balanced, linear cost-share equilibrium.*

In the special case where there is only one public good ($N = 1$), equations [6] (for $z^* > 0$) require that $a_i = 0$, $i = 1,..., M$, i.e., they imply cost-share systems of the form: $g_i(z) = b_i G(z)$, $i = 1,..., M$, $\Sigma_{i=1}^{M} b_i = 1$. This model coincides with Kaneko's (1977) «ratio equilibrium».

The correspondence established in Section 2.12 between linear cost-share equilibria and the common version of the Lindahl model allows for the following interpretation. Multiply the two terms of condition [5] by z_j^* and sum with respect to j. We then observe that a balanced, linear cost-share equilibrium with parameters a_{ij} and b_i corresponds to an equilibrium of the common version of the Lindahl model with personalized prices p_{ij} satisfying:

$$\Sigma_{j=1}^{N} p_{ij} z_j^* = \Sigma_{j=1}^{N} a_{ij} z_j^* + b_i \Sigma_{j=1}^{N} \Sigma_{h=1}^{M} p_{hj} z_j^*, \; i = 1,..., M.$$

Using the no-transfer condition $\Sigma_{j=1}^{N} a_{ij} z_j^* = 0$, the equality becomes:

$$\Sigma_{j=1}^{N} p_{ij} z_j^* = b_i \Sigma_{j=1}^{N} \Sigma_{h=1}^{M} p_{hj} z_j^*, \; i = 1,..., M,$$

i.e.:

$$b_i = \frac{\Sigma_{j=1}^{N} p_{ij} z_j^*}{\Sigma_{j=1}^{N} \Sigma_{h=1}^{M} p_{hj} z_j^*}, \; i = 1,..., M. \tag{7}$$

One can extend the argument in Section 2.9 above to N public goods and show that, by the first order conditions for individual utility maximization, $\Sigma_{j=1}^{N} p_{ij} z_j^*$ is the marginal valuation of the vector of public goods z^* by person i. This allows us to interpret the parameter b_i as:

$$b_i = \frac{\text{Marginal value of } z^* \text{ to } i}{\text{Social marginal value of } z^*}, \; i = 1,..., M.$$

This interpretation justifies the three following remarks. First, in a balanced, linear cost-share equilibrium, person i's contribution is $\Sigma_{j=1}^{N} a_{ij} z_j^* + b_i G(z^*) = b_i G(z^*)$, i.e., i pays a fraction of the cost of vector z^* that is proportional to her marginal valuation of z^*.

Second, the contributions agree with the benefit principle because:

$$\frac{i\text{'s contribution}}{h\text{'s contribution}} = \frac{b_i G(z^*)}{b_h G(z^*)} = \frac{b_i}{b_h} = \frac{i\text{'s marginal valuation of } z^*}{h\text{'s marginal valuation of } z^*}.$$

Finally, recall that, by [4], the parameter b_i corresponds to the profit share θ_i, which, in turn, can be interpreted as property rights over the input implicit in the technology. Because the technology is now publicly owned, we can say that a balanced, linear cost-share equilibrium assigns the rights to the usufruct of the implicit input among the different citizens in proportion to the marginal valuation of z^* by each person. Thus, the concept embodies a theory of the allocation of the benefits from publicly-owned resources.

As mentioned above, it is hard to find a real-world counterpart of the imaginary market proposed by the common version of the Lindahl model. But imagine (say, a public enterprise providing cable TV) that a public good is produced by using a publicly owned technology and is allocated through the market by means of personalized prices that consumers consider exogenous. Under decreasing returns to scale, the firm generates positive profits that are distributed among citizens in accordance with the coefficients θ_i. The concept of a balanced, linear cost-share equilibrium then offers a theory for the determination of θ_i in the terms of the marginal valuations of the good.

2.14. *Remarks*

A brief comment on the existence of balanced, linear cost-share equilibria is in order. Given that, contrary to the modern version of the Lindahl model, our concept does not require profit maximization, equilibrium may in principle exist under increasing returns to scale. But the basic existence theorem found in my article with Andreu Mas-Colell (1989) covers only economies where the functions $U_i(x_i, z)$ are concave and the cost function is convex (i.e., decreasing returns to scale). Otherwise, the existence issue boils down to comparing the curvatures of the indifference curves and the cost function. There is a special case in which the equilibrium exists no matter what the curvatures are, namely that of identical consumers (including the case $M = 1$), which illustrates that balanced, linear cost-share equilibria may exist under increasing returns to scale.

Finally, note that the notion of a balanced, linear cost-share equilibrium is formulated, in a relative generality, for economies with M persons and N public goods, and without restrictions on the utility functions. But we postulate a single private good that is used as input. Is it possible to extend the analysis to many private goods? The extension to produced private goods (perhaps subject to externalities) does not pose any problem: it can in fact be achieved through a simple reinterpretation of the goods. Section 3.5 below will present the details. Another relatively straightforward extension is the consideration of several private goods that are initial resources and that could be used as inputs. Evidently, one should then add the prices and supply and demand equations for private goods. An extension that appears difficult, however, is the inclusion of intermediate goods.

It is not appropriate to talk about public goods without mentioning, even if it is in passing, the implementation problem. As is well known, depending on the decision scheme, individual incentives may conflict with the common good, and in particular, with economic efficiency. One may formulate the implementation problem as follows: Can we design rules of the game such that the non-cooperative and selfish interaction of the individuals leads to the desired solution? There is an extensive literature on this subject, in particular for the common version of the Lindahl model: see, for example, Mark Walker (1981).

The work of Luis Corchón (1989), followed upon by Simon Wilkie (1989), offers a particularly intuitive mechanism to implement Kaneko's ratio solutions (which coincide, it should be recalled, with balanced, linear cost-share equilibria when there is a single public good). The idea is surprisingly simple: each citizen pledges a certain percentage of the cost of the public good, while simultaneously proposing an increase or decrease in the quantity of public good proposed by all the other citizens. One rule of the game is that the supply of the public good is zero if the sum of the planned cost percentages is less than one hundred. By definition, in a non-cooperative Nash equilibrium no citizen is interested in modifying the quantity of public good or the percentage of cost she contributes. Corchón shows that non-cooperative Nash equilibria coincide with Kaneko's solutions.

To conclude, let us return to Lindahl. Kaneko's ratio solutions and the generalization presented in my article with Andreu Mas-Colell (1989) are, in some way, more true to Lindahl than the version with personalized prices. The two notions coincide under constant returns to scale, where the issue of ownership of the technology is irrelevant. But they diverge when returns to scale are variable and therefore, property rights on the technology matter.

The ownership of the technology has a sharp interpretation under decreasing returns, namely the ownership of a special, nonmarketed resource. If the resource is privately owned, then the benefit principle may be combined with the private ownership rights in the common version of the Lindahl model, where the shares θ_i in the profits of the firms that produce the public goods are data.

But if, on the contrary, the resource that is at the root of the decreasing returns is publicly owned, then our analysis offers a theory of cost-sharing or, equivalently, a theory of the distribution of the profits of public enterprises. The theory is based on the benefit principle and the notion of unanimity proposed by Lindahl.

The necessity of a theory of distribution of profits and losses is felt even more strongly when returns to scale are increasing. On the one hand, it is then more difficult to interpret the private ownership of a technology. On the other, economic efficiency requires that the revenues from the firm's sales be less than its costs. The «ownership rights» over the firm are, in effect, the duty to cover a fraction of the losses. They reflect «forced ownership», because a citizen would prefer to be relieved from such a burden.

The discussions has so far been limited to firms that produce public goods. Section 3 will study the implications of this analysis for publicly owned firms that produce private goods.

3. Distributing the surplus from publicly owned technologies

3.1. The price as a public good

Consider a firm that produces and sells a private good at a uniform price. The price is the same for every buyer, and in this sense, the price is a public good. More specifically, the ability to buy any amount of output at a given unit price is a public good. How can we apply the previous discussion to this kind of public good? Recall that the analysis is based on the principles of unanimity and of benefit, and that it prescribes that the costs of supplying the public good should be distributed according to the quotient:

$$\frac{\text{Individual marginal valuation of the public good}}{\text{Social marginal valuation of the public good}}$$

I must define, in order to apply the analysis, the benefit that the public good yields to the individuals as well as the cost of supplying the public good. I adopt a model similar to that in Section 2.11, but in which the output of the firm, denoted y, is a private good.

Again, x is a private good, initially owned by individuals (person i initially owns the quantity ω_i) and used as input by the firm. It is also the numeraire good. The utility function of person i is $x_i + u_i(y_i)$, where y_i is her consumption of the good produced by the firm and x_i is her (final) consumption of the numeraire good. As in Section 2.4 above, utility functions are quasilinear. The cost function (i.e., the inverse production function) is denoted $C(y)$: in other words, the technology available is such that in order to produce y units of output one must employ $C(y)$ units of numeraire.

As is well known, under this utility function and the assumption of interiority, consumer demand depends only on the price and not on wealth.

Denote person i's demand function by $\widetilde{y}_i(p)$. By the first order conditions, the quantity demanded satisfies the equation:

$$u_i^{!}(y_i) = p, \qquad [8]$$

i.e., $\widetilde{y}_i(p)$ is the inverse of the $u_i^{!}(y_i)$ function.

How much does the consumer value the ability to buy any amount of output at price p? In other words, what is the maximum quantity, let us call it $v_i(p)$, that person i would be willing to pay for this ability? A natural response is to identify $v_i(p)$ with i's consumer's surplus at price p.

Let us elaborate. If consumer i's wealth is (in units of the numeraire) R_i, such an ability allows her to achieve the utility level $-p_i\widetilde{y}_i(p)+R_i+u_i(\widetilde{y}_i(p))$; if, on the contrary, the ability to buy is absent, then her utility level will simply be R_i.[3] Therefore, the maximum quantity that consumer i is willing to pay for the ability to buy at price p is $u_i(\widetilde{y}_i(p)) - p\widetilde{y}_i(p)$, i.e., it equals i's consumer surplus, denoted $v_i(p)$. Differentiating $v_i(p)$ and using [8], we obtain:

$$v_i^{!} = u_i^{!}\,\widetilde{y}_i^{!} - \widetilde{y}_i - p\,\widetilde{y}_i^{!} = -\widetilde{y}_i\,. \qquad [9]$$

Thus, the graph of $\widetilde{y}_i(p)$ is that of $-v_i^{!}(p)$.[4] Incidentally, this fact justifies the interpretation of the consumer surplus at p^* as the area between the demand curve and the horizontal line at height p^*, because, if p_0 is the minimun price for which demand is zero, then:

$$\int_{p_0}^{p^*} - v_i'(p)\,dp = -v_i(p_0) + v_i(p^*) = v_i(p^*).$$

Recall that one must use $C(y)$ units of numeraire in order to produce y units of output. What is the cost of offering the public good «ability to buy any quantity of output at the unit price p»? The firm offering it must produce $\widetilde{y}(p) := \Sigma_{i=1}^{M}\widetilde{y}_i(p)$ at a cost of $C(\widetilde{y}(p))$ units of numeraire[5]. But the sale of $\widetilde{y}(p)$ units at price p generates a revenue of $p\widetilde{y}(p)$ units of the numeraire, that has to be substrated from the previous amount. Therefore, «the cost» to the firm of selling at price p is:

$$G(p) := C(\widetilde{y}(p)) - p\widetilde{y}(p), \qquad [10]$$

i.e., the firm's profit with the sign reversed.

[3] Consumer i takes both the price p and her wealth R_i as given. Her wealth is, typically, the sum of her initial endowment of numeraire and any profit income. In the case of an increasing return firm pricing at marginal cost, the profit «income» may be negative, i.e., may be a lump-sum tax directed at subsidizing the firm's losses.

[4] Provided that the solution to the consumer's maximization is always interior, in particular, provided that either there are no nonnegativity constraints on the amount of numeraire or wealth is large enough.

[5] The symbol «:=» means «defined as».

Now we are ready to apply the concept of balanced, linear cost-share equilibrium presented in Section 2.13 above. An equilibrium of this type is a price p^* and a vector of cost-share parameters $(b_1,..., b_M)$ such that, for $i = 1,..., M$, p^* maximizes

$$v_i(p) - b_i G(p). \qquad [11]$$

3.2. Equilibrium in the model of the price as a public good

Let us postpone the certainly substantive question of the existence of equilibrium (it will be analyzed in Section 3.4 below) and let us assume, for the time being, that an equilibrium p^* exists. As seen in Section 2.13 above, the parameter b_i must satisfy at equilibrium the condition:

$$b_i = \frac{\text{Marginal value of the public good to } i}{\text{Social marginal value of the public good}}, \quad i = 1,..., M,$$

which now becomes:

$$b_i = \frac{v_i'(p^*)}{\Sigma_{h=1}^{M} v_h'(p^*)},$$

or, using [9]:

$$b_i = \frac{y_i}{y}, \quad i = 1,..., M, \qquad [12]$$

where $y_i = \tilde{y}_i(p)$ is the consumption of output by individual i and $y = \tilde{y}(p)$ is aggregate output. The interpretation is rather natural. The equilibrium cost-share parameters b_i, i.e., the ones inducing the unanimous agreement on the price p^*, are determined by the relative individual consumption. If i consumes twice as much as h, then $b_i = 2b_h$, i.e., i's contribution towards covering the cost $G(p^*)$ is twice that of h.

A second interpretation for the parameter b_i can be derived as follows. Person i takes advantage of her ability to buy at the equilibrium price p^* by purchasing $\tilde{y}_i(p^*)$ units of output in exchange for $p^*\tilde{y}_i(p)$ units of numeraire. On the other hand, she has to contribute the amount $b_i G(p^*)$ towards the cost $G(p^*)$. If she initially owned ω_i units of the numeraire, the quantity of the numeraire x_i that she ends up with is, using [10]:

$$\begin{aligned} x_i &= \omega_i - p^*y_i - b_i G(p^*) \\ &= \omega_i - p^*y_i - [C(y) - p^*y]y_i/y \\ &= \omega_i - p^*y_i + [p^*y - C(y)]y_i/y, \end{aligned} \qquad [13]$$

i.e., the resulting allocation is the one obtained when the firm sets the price p^* and person i receives as profit income the fraction $b_i = y_i/y$ of the profits that the firm obtains from selling output.

Let us examine the properties of the equilibrium allocation. Equation [13] can be rewritten:

$$x_i = \omega_i - C(y)\, y_i / y,$$

i.e., person *i's* net contribution of numeraire is:

$$\omega_i - x_i = C(y)\, y_i / y, \qquad [14]$$

which can be interpreted in two equivalent ways. In the first place [14] indicates that person *i's* net contribution of numeraire is equal to her consumption of output multiplied by the *average* cost. The market price p^* is equal to the marginal cost, but the net payment, once we take into account the profit income, agrees with the average cost.

Alternatively, it is clear that the sum of the final holdings of numeraire plus the amount of numeraire spent in production exhausts the aggregate initial endowments of numeraire, i.e.,

$$\Sigma_{h=1}^{M} x_h + C(\tilde{y}(p)) = \Sigma_{h=1}^{M} \omega_h,$$

which allows us to rewrite [14] in the form:

$$\frac{\Sigma_{h=1}^{M} (\omega_h - x_h)}{\Sigma_{h=1}^{M} y_h} = \frac{\omega_i - x_i}{y_i}, \text{ for every person } i, \qquad [15]$$

i.e., for each person, *the relative consumption of output is equal to the relative contribution of numeraire.* This property will be discussed in more detail below.

The equilibrium allocation has another interesting property. Recall from Section 2.11 that cost-share equilibria are Pareto efficient. In our case, this means that the resulting allocation is an optimun relative to the allocations where the marginal benefit of the public good (i.e., the marginal rate of substitution between the produced good and the numeraire) is the same for every consumer. Because this is a necessary condition for (absolute) Pareto efficiency, it follows that the resulting allocation is Pareto efficient. In particular, the price p^* has to equal the marginal cost C'.

3.3. Shareholder unanimity

Josep Farrel (1985) came to a similar conclusion starting from the problem of shareholder unanimity in a corporation. As mentioned in Section 2.3 above, economic analysis typically postulates that shareholders unanimously agree on profit maximization. But this assumption has the implicit connotation that shareholders do not consume the output of the firm. If the firm produces a private good that its shareholders acquire in the market at a uniform price, then the ultimate interests of a shareholder-consumer result from two opposing forces represented by her profit income and her consumer surplus. On one extreme, a shareholder-consumer who holds one share among thousands, but who spends a large part of her income purchasing the firm's product would prefer that output be practically free. On the other extreme, a shareholder who does not consume output would prefer the monopoly price. A shareholder-consumer's preferred price will lie between zero and the monopoly price, and it will typically be higher the larger her holding of shares is and the lower her consumption of output is. Given her share θ_i in the profits of the firm, shareholder *i's* preferred price is the one that maximizes

$$v_i(p)+\theta_i\,[p\,\tilde{y}(p)-C\,(\tilde{y}(p))].$$

Using [9], the first order condition of the maximization problem is:

$$-\tilde{y}_i+\theta_i\,[\tilde{y}+p\,\tilde{y}'-C'\,\tilde{y}']=0 \qquad [16]$$

Summing [16] with respect to i and remembering that, by definition, $\Sigma_{i=1}^{M}\theta_i = 1$, we obtain:

$$-\tilde{y}+\tilde{y}+(p-C')\,\tilde{y}'=0,$$

which implies (unless $\tilde{y}' = 0$ the efficiency condition:[6]

$$p = C',$$

a condition that, when applied to [16], gives us:

$$-\tilde{y}_i+\theta_i\tilde{y}=0,$$

i.e.,

$$\theta_i=\tilde{y}_i/\tilde{y},$$

as in [12]. In words, if a profit distribution exists for which all the shareholders prefer the same price, then that price is equal to the marginal cost, and person *i's* profit share coincides with her share in the consumption of output.

3.4. The price as a public good: the existence of equilibrium

Let us return to the case where all consumers are shareholders. As indicated in Section 3.1 above, the existence of equilibrium in the model of the price as a public good is not a trivial issue. The general existence theorem in my article with Andreu Mas-Colell (1989) requires convexity hypotheses, in particular the quasiconcavity of utility functions, which in the quasilinear case is equivalent to the concavity of the benefit functions. But if the price is seen as a public good, then the benefit that i derives from the public good is the consumer surplus $v_i(p)$. Is $v_i(p)$ a concave function? The answer is no. Indeed, using [9] we can compute:

$$v_i''(p)=\frac{d\,v_i'(p)}{dp}=-\tilde{y}_i'(p),$$

[6] Sufficient when the function C is convex: recall that in the previous section efficiency is obtained without any conditions on the function C. Also note that if $\tilde{y}_i'$ were zero, that is, if the demand function were totally inelastic, then the equality of price and marginal cost would no longer be a necessary condition for the efficiency of an interior allocation.

an expression that is positive because the demand curve has a negative slope here. Therefore, the existence theorem found in Mas-Colell and Silvestre (1989) is not applicable to the model of the price as a public good.

In fact, the equilibrium may not exist at all. Consider the following simple example of nonexistence. The average cost of production is constant and equal to one. The economy consists of two persons, with demand functions:

$$\tilde{y}_1(p) = 3 - 2p,$$
$$\tilde{y}_2(p) = 4 - p.$$

If an equilibrium exists, then the price has to be equal to marginal cost, i.e., $p^* = 1$, and, consequently, using [12]:

$$b_1 = \tilde{y}_1(1)/\tilde{y}(1) = (3 - 2) / (7 - 3) = 1/4,$$
$$b_2 = \tilde{y}_2(1)/\tilde{y}(1) = (4 - 1) / (7 - 3) = 3/4.$$

For the price $p^* = 1$ and the parameters $(b_1, b_2) = (1/4, 3/4)$ to constitute an equilibrium, the price $p^* = 1$ must maximize i's consumer surplus and the firm's profits multiplied by b_i, both for $i = 1$ and $i = 2$. For person 1, $p^* = 1$ has to maximize:

$$v_1(p) + (p - 1)(7 - 3p)/4. \qquad [17]$$

Does $p^* = 1$ maximize this expression? We can compute, using [9], that its first derivative is

$$-\tilde{y}_1(p) + (7 - 3p - 3p + 3)/4,$$

which if evaluated at $p^* = 1$, yields:

$$-(3 - 2) + (7 - 3)/4,$$

obviously equal to zero. But we should resist the temptation to prematurely conclude that $p^* = 1$ is a maximum. The first order conditions are only necessary, and are satisfied both at a maximum and at a minimum. If we proceed to compute the second derivative of [17], we obtain:

$$\frac{d}{dp}(-3 + 2p + [7 - 6p + 3]/4) = 2 - 6/4 > 0,$$

i.e., $p^* = 1$ is a local *minimum* of expression [17]. It follows that our candidate for the equilibrium does not satisfy the required conditions. Given that this was the only candidate, we conclude that no equilibrium exists in this example of the model of the price as a public good.

3.5. The allocation of output as a vector of public goods

Let us summarize the previous analysis. We have considered the price of a product as a public good the level of which is determined by the conditions of a balanced, linear cost-share equilibrium, conditions that reflect the benefit and unanimity principles. Assuming that an equilibrium exists, we obtain an allocation that has two interesting properties. First, Pareto efficiency. Second, the net contribution of numeraire by person i equals her consumption of output multiplied by the average cost (or equivalently, that her relative consumption of the output equals her relative contribution of numeraire).

Unfortunately, the equilibrium may not exist at all. We ask: can one obtain the same allocation as the equilibrium of a model with better existence properties? The answer, obtained in Mas-Colell and Silvestre (1991), is affirmative. The relevant equilibrium concept is, again, an application of the notion of balanced, linear cost-share equilibrium. Contrary to the previous application, which considered a one-dimensional public good, namely, the price of the output, I now apply it to a vector of public goods whose dimension is the number of individuals in the economy.

Specifically, I view «the output consumed by Ms. i», y_i, and the «output consumed by Mr. h», y_h, as distinct goods. The vector $(y_1,..., y_M)$ of the allocation of output among individuals is considered a vector of $N = M$ public goods: it has to be determined by unanimous agreement, and its cost has to be shared among individuals. Person i's preferences are formally defined over vectors $(x_i, y_1,..., y_M)$ and represented by the utility function $x_i + w_i(y_1,..., y_M)$. But, to the extent that there are no externalities in consumption, the function w_i is constant with respect to the variables subscripted by any $h \neq i$, i.e., one can write $w_i(y_1,..., y_M) = u_i(y_i)$. Similarly, the cost function is, in principle, a function G of M variables $y_1,..., y_M$, but in fact it only depends on their sum, i.e., $G(y_1,... y_M) = C(\Sigma_{i=1}^{M} y_i)$.

Recall from Section 2.12 that a linear cost-share equilibrium is characterized by parameters a_{ij} and b_i. Because $(y_1,..., y_M)$ is now viewed as a vector of public goods, the parameters a_{ij} and b_i have to be such that every person chooses the same vector $(y_1,...,y_M)$ when solving the program $(i = 1,..., M)$:

$$\max_{(y_1,...,y_M)} u_i(y_i) - \Sigma_{j=1}^{M} a_{ij} y_j - b_i\, C(\Sigma_{j=1}^{M} y_j). \qquad [18]$$

We saw in Section 2.12 that a linear cost-share equilibrium can be interpreted as an equilibrium of the common version of Lindahl's model. It is not difficult to verify that, in the present application, it can also be interpreted as an Arrow-Debreu equilibrium for an economy defined by a firm with production function C^{-1} and M consumers defined by the functions u_i, the initial endowments ω_i and by profit shares equal to the parameters b_i, i.e., where:

$$\theta_i = b_i. \qquad i = 1,..., M.$$

More precisely, if p is the equilibrium price of the output in the Arrow-Debreu economy, then the equilibrium parameters of the linear cost-share model for $i = 1,..., M$, are given by:

$$b_i = \theta_i,$$
$$a_{ii} = p\,(1 - \theta_i) \qquad [19]$$
$$a_{ij} = -\theta_i p,\ j \neq i.$$

It is easy, for instance, to check that the maximization program [18] is then equivalent to the program:

$$\max_{(y_1,\ldots,y_M)} u_i(y_i) - p\,(1 - \theta_i)\,y_i + \Sigma_{j=1,j\neq i}^{M} \theta_i\, p y_j - \theta_i C(\Sigma_{j=1}^{M} y_j),$$

or,

$$\max_{(y_1,\ldots,y_M)} u_i(y_i) - p y_i + \theta_i[p\Sigma_{j=1}^{M} y_j - C(\Sigma_{j=1}^{M} y_j)],$$

which is in turn equivalent to the programs of profit maximization, given p, and utility maximization, given p and consumer wealth. The latter are, of course, the maximization programs that define an Arrow-Debreu equilibrium.

Summarizing, if we apply the notion of linear cost-share equilibrium to the case in which the vector $(y_1,\ldots, y_M)$ of individual consumptions of output is considered as a vector of public goods, then we obtain an equilibrium allocation for an Arrow-Debreu economy in which person i receives the share $\theta_i = b_i$ of the firm's profits. Clearly, this allocation is Pareto efficient.

Let us proceed one step further and apply the notion of balanced, linear cost-share equilibrium to this case. We saw that the equilibrium satisfies the *no-transfer condition* [6]:

$$\Sigma_{j=1}^{N} a_{ij} y_j = 0, \qquad i = 1,\ldots, M,$$

that, together with [19], leads us to:

$$b_i = \frac{y_i}{\Sigma_{j=1}^{N} y_j},\ i = 1,\ldots, M,$$

which is precisely condition [12]. Therefore, we find again the equilibrium allocation of the model with the price as a public good discussed in Sections 3.1-3. As shown in Mas-Colell and Silvestre (1991), the equilibrium allocations of the two models coincide: more precisely, if we apply the notion of balanced, linear cost-share equilibrium to the model of the price as a public good, then we find the same allocations as when we apply the same notion to the model of the allocation of output among consumers as a vector of public goods[7]. In particular, the equilibrium allocations are Pareto efficient and satisfy condition [12] in either model.

[7] As long as all the functions are differentiable and the cost function is convex; if not the situation becomes more complex: Mas-Colell and Silvestre (1991) offers a detailed analysis.

The second model is more complex than the first, both interpretatively (the interpretation of the price as a public good is more direct) and formally (the second model requires the M^2 parameters a_{ij} in addition to the M parameters b_i). But the second model offers a theoretical advantage, namely, the existence of equilibrium is guaranteed under the usual convexity conditions.

Moreover, interpreting the allocation of output as a vector of public goods permits extending the analysis to cover externalities. If, for example, my consumption y_i of output affects the utility of other members of society, then y_i shows up as an argument in the utility functions of other individuals, and therefore, it is in fact a public good (or bad). The resulting equilibrium allocation is, of course, Pareto efficient, and it satisfies a condition similar to [12]. The details can be found in Mas-Colell and Silvestre (1991).

3.6. Three scenarios of publicly owned technologies

The two properties of Pareto efficiency and condition [12] have been derived for a variety of versions of an equilibrium model inspired by Lindahl. The two properties bestow concrete meaning to the notion that the technology is not privately owned but publicly owned. The two properties have been applied to a public firm, where they have implied a particular manner of distributing profits (or losses when returns to scale are increasing). My next objective is to study other properties of the final allocation naturally associated with the idea of public ownership of the technology, as well as other scenarios in which technology is publicly owned[8].

So far I have only considered the scenario of a publicly owned firm, which displays the two following characteristics.

a) Social production process: no particular unit of output is produced by a particular worker; rather, all output is produced collectively by the firm's workers.

b) Market allocation process: the firm buys in the labor market and sells in the output market.

The resulting allocations display two main properties: Pareto efficiency, which in turn implies marginal cost pricing, and the equality of profit shares to shares in consumption.

A second scenario, that of a cooperative, is suggested by the personal experience of my colleague John E. Roemer, co-author of the research on which the rest of this text is based[9]. A group of friends jointly own a vine-

[8] I will only consider properties of the final allocation, i.e., I will abstract from behavioral or procedural aspects.

[9] See Roemer and Silvestre (1987, 1989, 1993).

yard, in which they invest labor time and from which they obtain wine for their personal consumption. As in the public enterprise, the production process is collective, but the allocation is not implemented through markets: there are no purchases, sales, prices or profits.

The third scenario that I consider is that of a common-pool resource. For example, people who live on the shores of a lake fish and consume their catches. Here production is individual: each fisher goes out to fish and comes back with a certain amount of fish aboard her vessel. I assume that all fishers are equally capable and lucky, so that if a fisher spends twice as much time as another one, then she catches twice as much fish as the other one. But the catch per hour of an individual fisher may depend on the total number of active fishers. In this case, the activity of any fisher imposes a negative external effect on the other fishers. Note that, as in the cooperative vineyard, no markets intervene in the allocation of fish.

The three scenarios can be modeled in the same manner. Let us consider again a society with M individuals and two goods: a private good, denoted y, produced with the help of a publicly owned technology, and a numeraire good, denoted x, which is initially owned by individuals (individual i initially owns the quantity ω_i) and used as input in the production. Here we interpret the numeraire good as labor; given the initial endowment ω_i and final consumption x_i, measured in hours of labor, we denote by $L_i := \omega_i - x_i$ the hours of work contributed by person i. We will also write $L := \Sigma_{i=1}^{M} L_i$.

We relax quasilinearity, and simply assume that person i's preferences are represented by the differentiable and increasing utility function $U_i(x_i, y_i)$. It will be more convenient here to represent the technology by a production function $f(L)$ instead of the cost function $C(y)$ used until now. The two formulations are equivalent because one function is simply the inverse of the other one, i.e., $y = f(L)$ if and only if $L = C(y)$. But the production function formulation allows us also to consider the extreme case in which the good y cannot be produced but is a publicly owned private good initially available in μ units. This extreme case, which we call «the manna economy», can be captured by writing $f(L) = \mu$, for all $L \geq 0$.

3.7. *The natural allocation when returns to scale are constant*

Let us provisionally impose the additional assumption that the technology displays constant returns to scale, i.e., for a certain $\beta > 0$, and for all $L > 0$,

$$f(L) = \beta L$$

We ask: what allocation of the two goods best reflects the dual structure of ownership, namely, the private ownership of the numeraire together with the public ownership of the technology? Consider, for $i = 1,..., M$, the program:

$$\max U_i(x_i, y_i) \text{ subject to } y_i \leq \beta\,(\omega_i - x_i), \qquad [20]$$

which, under the assumption that U_i is strictly quasiconcave, has a unique solution, denoted (x_i^*, y_i^*). Because of monotonicity, the solution has to satisfy the constraint with equality, i.e.,

$$x_i^* = \omega_i - y_i^* / \beta \qquad [21]$$

If, moreover, the solution is positive, then the first order conditions imply that the individual marginal rates of substitution are equal to each other and to the marginal rate of transformation, i.e.,

$$\frac{\dfrac{\partial U_i(x_i, y_i)}{\partial y_i}}{\dfrac{\partial U_i(x_i, y_i)}{\partial x_i}} = \frac{1}{\beta}, \qquad [22]$$

which in turn is equivalent to Pareto efficiency[10].

I will argue that the allocation $((x_1^*, y_1^*), ..., (x_M^*, y_M^*))$, in addition to satisfying the desirable efficiency property, naturally reflects the public ownership of the technology accompanied by the private ownership of the numeraire. First, I interpret this allocation in the three scenarios just described.

In the scenario of the public enterprise, $((x_1^*, y_1^*), ..., (x_M^*, y_M^*))$ is the allocation that results when the firm sets the output price at $p = 1/\beta$ (the price of the numeraire is obviously unity) and consumers freely choose the quantity of output to buy and the amount of numeraire to sell. Given that profits are zero, condition [21] is nothing but the budget equality of person i.

In the scenario of the cooperative vineyard, $((x_1^*, y_1^*), ..., (x_M^*, y_M^*))$ is the allocation obtained when the rules are such that each person can freely choose her contribution of labor hours knowing that output will be distributed in proportion to the hours worked.

Finally, in the scenario of the lake, the technology is such that the catch per hour is β units, independent of the number of fishers on the lake, i.e., no external effects are generated among fishers. The allocation $((x_1^*, y_1^*), ..., (x_M^*, y_M^*))$ results when each fisher independently chooses her fishing hours and consumes her catch.

In each of these scenarios, the allocation $((x_1^*, y_1^*), ..., (x_M^*, y_M^*))$ is, intuitively, the one that best captures the idea that the production technology is publicly owned while the numeraire (labor) is privately owned. In particular, the individuals freely use the publicly owned technology without interpersonal transfers of the privately owned good. Besides Pareto efficiency, one can distinguish the following properties of the allocation $((x_1^*, y_1^*), ..., (x_M^*, y_M^*))$, which capture intuitive aspects of the idea that the technology is publicly owned while labor is privately owned.

[10] Of course, provided that in any efficient allocation production is positive and at least one consumer consumes positive amounts of both goods.

a) THE CONSUMPTION OF OUTPUT IS PROPORTIONAL TO THE CONTRIBUTION OF NUMERAIRE

We had obtained this property, expressed by [12], in the balanced, linear cost-share equilibrium models. Recall that, in the scenario of the public firm, it meant that the net contribution of numeraire by each person equals her consumption of output evaluated at average cost. Because returns to scale are now constant, the average cost is equal to the marginal cost, which is the price charged by the public firm.

Equality [21] ensures that, at the allocation $((x_1^*, y_1^*),..., (x_M^*, y_M^*))$, the contribution of numeraire per unit consumed is equal to the average cost. In the vineyard and lake examples, [21] can be interpreted as «to each according to his work.»

b) EQUAL BENEFIT FROM THE PUBLICLY OWNED TECHNOLOGY

If the publicly owned technology did not exist, then, leaving aside conceivable redistributions of numeraire, person *i's* final consumption vector would equal her vector of initial endowments $(0, \omega_i)$. On the other hand, the presence of the publicly owned technology allows each person to achieve a final consumption vector different from her initial endowments. For example, person i reaches the final consumption vector $(x_i, y_i) = (x_i^*, y_i^*)$ in the allocation $((x_1^*, y_1^*),..., (x_M^*, y_M^*))$. Given that all persons own the technology to the same extent, it would be desirable that they also benefit from it to the same extent. One way of measuring the benefit that each person derives from the public ownership of the technology is to compare the final consumption vector (x_i, y_i) that she actually obtains with the autarkic vector $(\omega_i, 0)$ that she would end up with in the absence of the technology. The difference between the two vectors is $(x_i - \omega_i, y_i)$. Its first component represents the change in the consumption of the numeraire, and is typically negative, whereas its second component, representing the change in the consumption of the produced good, is positive. We have observed that equality [22] is satisfied at any Pareto efficient allocation, i.e., the marginal rates of substitution are equal to each other and to the marginal rate of transformation. Therefore, it is natural to evaluate one unit of the produced good at $1/\beta$ units of the numeraire. The value, so computed, of the difference vector $(x_i - \omega_i, y_i)$ is $x_i - \omega_i + y_i/\beta$. The idea that all the individuals equally benefit from the technology that they jointly own is the condition that the difference vectors corresponding to distinct individuals have the same value, i.e.,

$$x_i - \omega_i + y_i/\beta = x_h - \omega_h + y_h/\beta \text{ for all individuals } i, h \qquad [23]$$

Clearly, the allocation $((x_1^*, y_1^*),..., (x_M^*, y_M^*))$ satisfies [23], because, by [21], we have that $x_i^* - \omega_i + y_i^*/\beta = 0$, for all i.

c) TECHNOLOGICAL MONOTONICITY

A solution satisfies technological monotonicity when it yields increased welfare to every citizen whenever the technology improves. The property was introduced by John Roemer (1986) and has played a prominent role in axiomatic social choice theory: see, e.g., Hervé Moulin (1988) and the references therein. It is, at first blush, a desirable property in our one input and one output prototype, although, as we shall discuss in Section 3.15 below, not under closer scrutiny. In any event, it is clear that the allocation $((x_1^*, y_1^*),..., (x_M^*, y_M^*))$ does satisfy it. Indeed, assume that due, for example, to technological progress, the parameter β increases, i.e., the publicly owned technology becomes more productive. Then the solution to the program [20] will change. More specifically, the set of points (x_i, y_i) that satisfy the restriction «$y_i = \beta\,(\omega_i - x_i)$» will become larger, and the utility attained by person i will increase ($i = 1,..., M$). In other words, if the publicly owned technology improves, the utility of every consumer i will increase at the solution to the problem max $U_i(x_i, y_i)$ subject to $y_i \leq \beta\,(\omega_i - x_i)$.

Summarizing, if the technology displays constant returns to scale, then the allocation $((x_1^*, y_1^*),..., (x_M^*, y_M^*))$, defined by the condition that, for each person i, (y_i^*, x_i^*) solves the maximization program [20], can easily be interpreted in the three scenarios considered. One can say that, in each situation, the allocation $((x_1^*, y_1^*),..., (x_M^*, y_M^*))$ simultaneously upholds the public nature of the ownership rights over the technology together with the private character of the ownership rights over the productive input. In addition to Pareto efficiency, it also satisfies the interesting properties of proportionality of individual consumption to the individual contribution of numeraire, and equal benefit from the public ownership of the technology. Thus, the allocation $((x_1^*, y_1^*),..., (x_M^*, y_M^*))$ can be considered as the *natural solution* when returns to scale are constant and the technology is publicly owned while the productive input is privately owned.

3.8. Variable returns to scale

Is it possible to extend the natural solution to variable returns? More precisely, are there allocations which, in addition to being Pareto efficient, satisfy the conditions of consumption proportionality and equal benefit?

A previous task is to define the properties for economies with variable returns to scale. Consideration is restricted to Pareto efficient allocations.

Consumption proportionality is expressed by the, by now familiar, equality [15].

The *equal-benefit* property is defined by condition [23] generalized so that it applies to variable returns. To this end, write p instead of $1/\beta$, where p is defined as:

$$p = \frac{\dfrac{\partial U_i(x_i, y_i)}{\partial y_i}}{\dfrac{\partial U_i(x_i, y_i)}{\partial x_i}}, \qquad [24]$$

for any person i who consumes positive quantities of the two goods in the allocation that is being considered: the efficiency assumption implies that p is well defined. The equal-benefit condition is then written:

$$x_i - \omega_i + py_i = x_h - \omega_h + py_h, \text{ for every pair of individuals } i, h \quad [25]$$

Are there allocations that, in addition to being Pareto efficient, satisfy the two conditions? The general answer is negative: such allocations only exist in particular cases. Of course, one such case is that of constant returns to scale. A second special case where the two conditions (plus efficiency) are compatible is that of identical individuals, i.e., $U_i = U$, $i = 1,..., M$ (the preferences of the M individuals are equal) and $\omega_i = \overline{\omega}$ (the initial endowments of the M individuals are equal). The «natural» allocation is now $((x^*, y^*),..., (x^*, y^*))$, where (x^*, y^*) solves the program:

$$\max U(x, y) \text{ subject to } y \leq f(M[\overline{\omega} - x])/M.$$

The allocation $((x^*, y^*),..., (x^*, y^*))$ trivially satisfies the conditions of consumption proportionality and equal benefits. On the other hand, any improvement in technology expands the constraint set, and, therefore, cannot decrease utility.

But, in general, if we maintain the property of Pareto efficiency, then the two conditions are incompatible, and so is technological monotonicity with either one of the two: Section 3.14 below offers some illustrative examples. It follows that, in general, no allocation will capture the property relations considered here as clearly as the natural solution does for the special cases of constant returns or identical persons. If we maintain the efficiency condition, then it will be necessary to select only one additional property.

The conditions of efficiency and consumption proportionality (equality [15]) define the solution that we had already reached through the equilibrium conditions in Sections 3.1-3 and 3.5. It will now be called the *proportional solution.* Efficiency and the equal-benefit condition (equality [25]) define a second solution, to be called the *equal-benefit solution.*

At first glance, it is not obvious that, in general, a solution exists that satisfies efficiency and technological monotonicity. There is, in fact, one, presented a few years ago by Andreu Mas-Colell (1980) in a different context (actually, as a tool to show that a certain core was nonempty) and discussed in greater detail in Moulin (1987) and Moulin and Roemer (1989). It is known as the *constant-return-equivalent solution* (or the linear equivalent solution). Its interpretation is somewhat complex: the reader may consult any of these references for a detailed account. Formally, it is defined by Pareto efficiency and the condition:

«there exists a number $\overline{\beta} > 0$ such that, for every individual i,

$$U_i(x_i, y_i) = V_i(\overline{\beta}, \omega_i)»,$$

where V_i is i's indirect utility function[11]. Intuitively, the allocation is such that each person is indifferent between the consumption vector that she gets and the one that she would get by the maximization program [20] if she had access to an imaginary linear technology defined by a parameter β (same β for every person).

3.9. The manna economy

The application of the proportional solution and the equal-benefit solution to the extreme case of the manna economy presented earlier (end of Section 3.6) will help clarify their meaning. Recall that, in the manna economy, good y is private and nonproducible: it is initially available in the fixed amound μ (fallen from the sky). Therefore, any feasible allocation in this economy is a redistribution of the initially available amounts of the two goods $\omega = \Sigma_{i=1}^{M} \omega_i$ and μ. Obviously, given that the two goods are desirable, at any efficient allocation the amounts that people receive should, in the aggregate, exhaust the available quantities of the two goods. Attention is thus restricted to allocations $((x_1, y_1),...,(x_M, y_M))$ satisfying:

$$\Sigma_{h=1}^{M} x_h = \Sigma_{h=1}^{M} \omega_h, \quad \Sigma_{h=1}^{M} y_h = \mu. \qquad [26]$$

Consider first the proportional solution, defined by efficiency plus equality [15] i.e.,

$$\frac{\Sigma_{h=1}^{M} (\omega_h - x_h)}{\Sigma_{h=1}^{M} y_h} = \frac{\omega_i - x_i}{y_i} \quad \text{for every person } i,$$

which, when [26] is satisfied, can be rewritten as:

$$0 = \omega_i - x_i.$$

In other words, the proportional solution is, in the manna economy, a Pareto efficient allocation where each person consumes her initial endowment of the numeraire. If $M = 2$, then the allocations that satisfy equality [26] can be represented as points of an Edgeworth Box of dimensions $\mu \times \omega$: see Figure 4. The proportional solution is the point on the contract curve (the locus of Pareto efficient allocations) where the allocation of numeraire is (ω_1, ω_2). The interpretation is clear: the proportional solution forbids any reduction in the quantity of numeraire initially owned unless it is related to production costs. But manna falls from the sky at zero cost: thus, each person can freely consume all her initial endowment of numeraire.

The manna economy highlights the contrast between the proportional

[11] Recall that i's indirect utility function is defined as:

$$V_i(p, R_i) = U_i(\tilde{y}_i(p, R_i), \tilde{x}_i(p, R_i)),$$

where $(\tilde{x}_i(p, R_i), \tilde{y}_i(p, R_i))$ is i's demand vector, i.e., the solution to:

$$\max U_i(x_i, y_i) \text{ subject to } py_i + x_i = R_i.$$

solution and the equal-benefit solution. Recall that the latter is defined by efficiency and equality [25] which equality can be rewritten as:

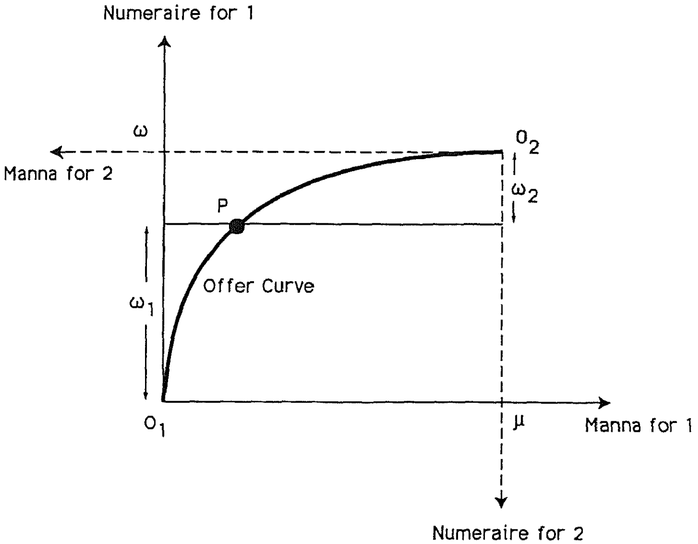

Figure 4
The Proportional Solution in the Manna Economy is Point P

$$x_i - \omega_i + py_i = \Sigma_{h=1}^{M} (x_h - \omega_h + py_h)/M, \text{ for every person } i,$$

or, using [26],

$$x_i + py_i = \omega_i + p\mu/M, \text{ for every person } i,$$

i.e., the proportional solution is a Pareto efficient allocation in which the value of the consumption vector of person i (evaluated at the price determined by the marginal rate of substitution) equals the value of her initial endowment of numeraire plus one *M-th* of the public endowment of manna. We may visualize the allocation in the following two-step manner. First, the manna is privatized in equal parts, so that person i owns the vector $(\omega_i, \mu/M)$. A competitive market is then established. Each person maximizes her utility at the equilibrium price p subject to her budget constraint, namely, subject to the constraint that the value of her consumption must not exceed her wealth $\omega_i + p\mu/M$. See Figure 5.

The two solutions sharply contrast. Imagine that the first person derives practically no utility from manna (her indifference curves are almost vertical, even for very small amounts of manna), while manna is indispensable for the second person. Both the proportional solution and the equal-benefit solution give almost all the manna to the second person[12]. In the pro-

[12] Pareto efficiency, which both solutions satisfy, requires that all the manna end up in the hands of the second person in the extreme case where the first person has no use for manna.

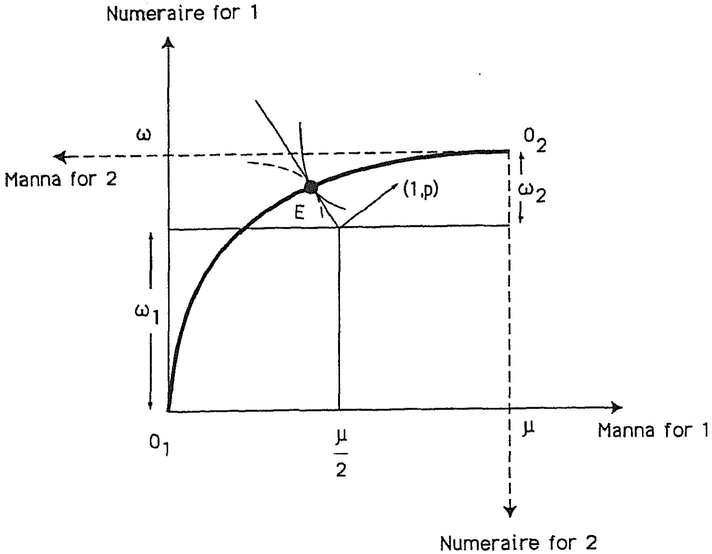

Figure 5
The Equal Benefit Solution in the Manna Economy is Point E.

portional solution the second person pays no numeraire. In the equal-benefit solution, on the contrary, the second person has to transfer a large quantity of numeraire to the first person because she has to buy the rights over (approximately) half of the manna fallen from the sky. The first, person benefits from the manna basically because she can exclude other people from using some fraction of the manna. The equal-benefit solution seems to reflect egalitarian private property rather than public ownership. This line of argument, developed in more detail in Roemer and Silvestre (1989), leads us to choose the proportional solution as a conceptually more adequate representation of the dual property structure. The following section applies these ideas to the three scenarios discussed earlier.

3.10. The proportional solution and the equal-benefit solution in the public firm

We leave the manna economy and return to the initial assumption of a produced private good: the publicly owned technology is represented by an increasing function $f(L)$. The conditions of physical feasibility and full utilization, expressed by equation [26] in the manna economy, can now be written:

$$\Sigma_{h=1}^{M} y_h = f(\Sigma_{h=1}^{M} \omega_h - \Sigma_{h=1}^{M} x_h). \qquad [27]$$

As we saw in previous sections (3.2,3,6), the proportional solution distributes the firm's profits in proportion to consumption. In fact, the proportional solution is defined by efficiency and equation [15], i.e.:

$$\frac{\Sigma_{h=1}^{M}(\omega_h - x_h)}{\Sigma_{h=1}^{M} y_h} = \frac{\omega_i - x_i}{y_i}, \text{ for every person } i,$$

or, using [27], and writing $L_h = \omega_h - x_h$, $h = 1,\ldots, M$, and $L := \Sigma_{h=1}^{M} L_h$,

$$py_i = pf(L)\, L_i/L = L_i + (pf(L) - L)\, L_i/L,$$

which is nothing but the budget equality of a consumer whose share in the firm's profits is θ_i when θ_i happens to be equal to L_i/L (or to $y_i/f(L)$: the two expressions coincide by [15] and [27]).

The equal-benefit solution, on the other hand, is defined by efficiency and equation [25], i.e.,

$$x_i - \omega_i + py_i = x_h - \omega_h + py_h, \text{ for every pair of individuals } i, h,$$

which, by [27], can be rewritten as:

$$-L_i + py_i = \Sigma_{h=1}^{M}(-L_h + py_h)/M = (-L + pf(L))/M, \text{ for every person } i, \quad [28]$$

which is nothing but the budget equality of a person who receives a $1/M$ share of the firm's profits

Observe that Pareto efficiency requires the firm to set its price equal to the marginal cost, and, thus, to suffer losses when returns to scale are increasing. Losses are, in the proportional solution, distributed among the consumers of output in proportion to the amount consumed. As we saw, this implies that the net amount paid by each consumer, taking into account the expenses incurred by the purchase of output in the market and the distribution of losses, equals her consumption of output evaluated at average cost. The equal-benefit solution, on the other hand, uniformly distributes the losses derived from marginal cost pricing, regardless of consumption. A person who consumes no output must nevertheless cover a $1/M$ share of the losses of the public firm. Of course, this person would prefer that the public firm did not exist. In this sense, the equal-benefit solution means forced ownership rather than public ownership when returns to scale are increasing.

3.11. The proportional solution and the equal-benefit solution in the cooperative

The proportional solution has here a direct interpretation, given by rewriting equation [15] as:

$$y_i/f(L) = L_i/L \quad [29]$$

i.e., the quantity of output received by a member of the cooperative is proportional to her contribution of labor. As indicated in Section 3.7, the equality expresses the principle «to each according to his work».

How should we interpret the equal-benefit solution? Equation [28] can be rewritten as:

$$py_i = L_i + (pf(L) - L)/M,$$

or, dividing through by $pf(L)$,

$$y_i/f(L) = L_i/pf(L) + 1/M - L/Mpf(L)$$

which, defining $\sigma = L/pf(L)$, can be written:

$$y_i/f(L) = \sigma L_i/L + (1 - \sigma)/M \qquad [30]$$

A similar expression, but with σ as an exogenous parameter, has been used by L. Dwight Israelsen (1980) and Amartya Sen (1966) in the classification of cooperative institutions. The $\sigma = 1$ case defines the «collective», which corresponds exactly to the proportional solution (equations [29] and [30] coincide.) The $\sigma = 0$ case defines the «commune», where output is distributed in equal shares, independent of labor contributions. Expression [30] suggests interpreting the equal-benefit solution as an intermediate case between the collective and the commune.

3.12. The proportional solution and the equal-benefit solution in the commons

Recall that fisher i gets $L_i f(L)/L$ units of output. If f displays decreasing returns to scale, then a fisher's activity generates negative external effects on the others. Let fisher i unilaterally choose the quantity L_i, taking L as given and without considering the external effect she creates[13]. Formally, let fisher i, $i = 1,..., M$, solve the problem:

$$\max_{L_i} U_i(\omega_i - L_i, L_i f(L)/L \qquad [31]$$

The resulting allocation is Pareto inefficient. The inefficiency, which takes the form of excess fishing, is described in a dramatic way by the expression «the tragedy of the commons», due to Garrett Hardin (1968).

A possible efficiency-inducing mechanism is the imposition of quotas or quantity limitations on the number of hours that a fisher may fish. For instance, each fisher is given, free of charge, a certain number of nontransferable permits; one permit authorizes fishing for an hour, but it is personalized and cannot be sold[14]. If the distribution of permits is appropriately chosen so that the resulting allocation is efficient, then such an allocation is the proportional solution, because it obviously satisfies condition [15].

On the other hand, the equal-benefit solution is equivalent to an egalitarian initial distribution of the fishing permits by which each fisher recei-

[13] Here we assume that fisher i, by taking L as given, ignores the influence of L_i on L. Alternatively, one can assume that she takes as given the hours of fishing by others, i.e., $\Sigma_{h \neq i} \psi L_h$. If M is large and each fisher is small, then the two formulations are similar. They are both inefficient under nonconstant returns.

[14] Equivalently, the quotas may apply to the amount of fish caught by a fisher.

ves L/M permits, followed by a competitive «secondary» market in permits. The total number of permits, L, has to be calculated in such a way that, once the market has redistributed the permits, the allocation is efficient. The equivalence can be argued in the following manner.

Denote the equilibrium price of a permit by ψ, measured in units of fish (i.e., the permits are paid for in fish, maybe smoked). If fisher i fishes for L_i hours, she has to hold L_i permits. Because she initially has L/M permits, her net sale (positive or negative) of permits is $L/M - L_i$. If this quantity is positive, then fisher i is a net seller of permits, and the sale would bring her $\psi\,[L/M - L_i]$ units of fish. If it is, on the contrary, negative, then i is a net buyer of permits, and the purchase costs her $-\psi\,[L/M - L_i]$ units of fish. Her final consumption of fish equals, in any event, her catch, $L_i f(L)/L$, plus the value of her net sales of permits, i.e.:

$$y_i = L_i f(L)/L + \psi\,[L/M - L_i]. \qquad [32]$$

What is the equilibrium price ψ of a permit? In equilibrium, given ψ and L/M, fisher i ($i = 1,\ldots, M$) solves:

$$\max_{L_i} U_i\,(\omega_i - L_i,\ L_i f(L)/L + \psi\,[L/M - L_i].$$

The first order condition of the program (which has to be satisfied if the solution is interior) is:

$$-\frac{\partial U_i(x_i, y_i)}{\partial x_i} + \frac{\partial U_i(x_i, y_i)}{\partial y_i}\,\frac{f(L)}{L} - \psi\frac{\partial U_i(x_i, y_i)}{\partial y_i} = 0, \qquad [33]$$

which, dividing by $\frac{\partial U_i\,(x_i, y_i)}{\partial x_i}$ and recalling [24], can be written:

$$-1 + pf(L)/L - p\psi = 0,$$

i.e.,

$$\psi = [pf(L) - L]\,/\,pL \qquad [34]$$

an expression that has the following interesting interpretation: «the price of the permit is the real (i.e., measured in units of output) profit per hour of labor that a private owner of the fishery would obtain in a competitive market system». The total value of the permits, ψL, corresponds then to total profits.

An alternative interpretation is traditional. Because Pareto efficiency requires that $1/p = f'(L)$, expression [34] can also be written: $\psi = f(L)/L - f'$, i.e., the price of the permit is the difference between the average product of labor and the marginal product of labor. When deciding how many hours to fish, a fisher takes into account both the physical product of her labor and the economic effects of the transfer (purchase or sale) of permits.

In this manner, she takes into account the social marginal product of her labor, including her influence on the productivity of other fishers, and her decisions lead to efficiency. When the fisher solves program [31], on the contrary, she only takes into account the social average product, which leads to an inefficient allocation.

Substituting [34] into [32], we get:

$$\begin{aligned} y_i &= L_i f(L)/L + (f(L)/L - 1/p)\,(L/M - L_i) \\ &= L_i f(L)/L + f(L)L/LM - f(L)\,L_i/L - L/pM) + L_i/p) \\ &= L_i/p + f(L)/M - L/pM, \end{aligned}$$

i.e.,

$$py_i = L_i + (pf(L) - L)/M, \qquad i = 1,\ldots, M,$$

which is precisely equation [28]. Equation [28] plus efficiency define the equal-benefit solution.

Permit systems, whether transferable or not, are frequently used to prevent overfishing. Pigovian tax-subsidy schemes, where a fisher is free to decide how many hours to fish, but has to pay a tax t per hour of fishing, are alternative mechanisms[15]. The tax is paid in fish. A tax-subsidy agency collects the taxes and distributes the receipts as lump-sum subsidies: fisher i takes both the tax rate t and her subsidy, denoted S_i, as given. If fisher i fishes for L_i hours, then the final quantity y_i available to her equals her catch, $L_i f(L)/L$, plus the subsidy S_i, minus the tax tL_i, i.e.:

$$y_i = L_i f(L)/L + S_i - tL_i.$$

The proportional solution has to satisfy [15], i.e.:

$$y_i = L_i f(L)/L.$$

Therefore, the resulting allocation (which we assume efficient) will be the proportional solution if:

$$S_i = tL_i,$$

i.e., if each fisher receives as subsidy precisely the amount that she paid as tax.

In order to characterize the equal-benefit solution in the Pigovian context, consider the maximization program of fisher i (1 = 1,.., M):

$$\max_{L_i} U_i(\omega_i - L_i, S_i + L_i f(L)/L - tL_i).$$

The first order condition is:

[15] Inspired by Alfred C. Pigou (1920).

$$-\frac{\partial U_i(x_i, y_i)}{\partial x_i} + \frac{\partial U_i(x_i, y_i)}{\partial y_i}\frac{f(L)}{L} - t\frac{\partial U_i(x_i, y_i)}{\partial y_i} = 0,$$

which is precisely equation [33], but with the tax rate t instead of the permite price ψ as unknown. Therefore, t is given by the right-hand side of [34]. In words, the Pigovian tax rate coincides with the price of the permit and can be interpreted in the same manner.

The equal-benefit solution must satisfy [28], i.e.,:

$$py_i = L_i + (pf(L) - L)/M, \text{ for every person } i.$$

It follows that, in the equal-benefit solution, fisher i should receive the subsidy:

$$S_i = (f(L) - L/p)/M = (f(L)/L - 1/p)L/M = t\, L/M \qquad i = 1,..., M,$$

(by [34] for $\psi = t$) i.e., the tax receipts tL should be equally distributed among the fishers.

3.13. Common pool resources and implicit inputs

Recall from Section 2.7 that profits manifest the price of the implicit input in competitive, privately owned firms subject to decreasing returns to scale. The implicit input is often interpreted, as in McKenzie (1959), as a human resource owned in various amounts by a group of individuals.

But the input implicit in a technology may, in some contexts, be a publicly owned natural resource rather than a privately owned human resource. In fact, a common pool resource, like the fishery, fits well with the notion of an input that is implicit in a decreasing returns to scale technology.

Imagine that the lake and its fish are privatized. For example, the law gives exclusive fishing rights to a particular person, who then hires fishers, pays them a wage and sells them the catch in a competitive market. The owner's profits are then the value of a natural resource rather than the return to a human resource.

Both the transferable permit scheme and the Pigovian tax-subsidy scheme assign a value to the publicly owned resource which corresponds to the profits of a privatized and commercialized lake. That value is, when permits are transferable, the total valuation of the fishing permits at the equilibrium price ψ. The value of the resource is given, under a Pigovian tax-subsidy scheme, by the aggregate tax receipts[16]. The proportional solution and the equal-benefit solutions distribute the returns to the natural resource among individuals in the same way as they distribute the profits of a

[16] There is an irrelevant change of units: as mentioned, I have measured permits and taxes in units of fish, while profits are typically measured in units of numeraire.

public firm. The proportional solution distributes the value of the resource in accordance with the individual use of output (or contribution of numeraire: the two criteria coincide), whereas the equal-benefit solution distributes the value among the population equally, independently of the individual consumption of output or contribution of numeraire. Again, the equal-benefit solution seems to reflect an egalitarian private ownership of the natural resource, rather than public ownership.

3.14. Examples

Section 3.8 asserted, without proof, that, as long as the condition of efficiency is maintained, it is in general possible to satisfy only one of the properties: *a)* proportionality of consumption to the contribution of numeraire; *b)* equal benefit from public ownership, or *c)* technological monotonicity. The proportional solution has been in fact defined by the properties: «efficiency + *a)*» and the equal-benefit solution by «efficiency + *b*»*)*. The discussion in the preceding sections shows that the proportional and equal-benefit solutions are typically different, showing that, under efficiency, *a)* and *b)* are in general incompatible.

The following example shows that, if the efficiency condition is maintained, then *b)* and *c)* are also incompatible in general. Let the economy consist of two people and a public firm, and consider the limit case where the first person is only interested in the numeraire good, i.e., $U_1(x_1, y_1) = x_1$[17]. If the production function is strictly concave, then profits Π are positive and distributed fifty-fifty in the equal-benefit solution. The final consumption vector of the first person is then $(\omega_1 + \Pi/2,\ 0)$, where ω_1 is her initial endowment of numeraire. Assume now that the production function is transformed into a linear function that yields a larger quantity of output for any positive quantity of the input. This is a better technology, but now the welfare of the first person worsens because profits are zero and, thus, her final consumption vector is $(\omega_1,\ 0)$. Therefore, the equal-benefit solution does not satisfy condition *c)* here.

A second extreme case shows that the proportional solution does not, in general, satisfy *c)*. The first person's utility function is $U_1(x_1, y_1) = x_1 + y_1$, and the second person's is $U_2(x_2, y_2) = y_2$, while the initial endowments of numeraire are $(\omega_1, \omega_2) = (1,1)$. Initially, the production function is $y = 5L/4$ for $L \leq 9/4$. The price of output is $p = 4/5$ and profits are zero: the consumption vectors are, thus, $(x_1, y_1) = (x_2, y_2) = (0,\ 5/4)$, i.e., $U_1(x_1, y_1) = 5/4$. Now the production function changes to:

$$f(L) = \begin{cases} 18L/4, \text{ for L} < 1/2, \\ 8/4 + L/2, \text{ for } L \text{ between } 1/2 \text{ and } 9/4, \end{cases}$$

[17] Neither this example nor the following one satisfy all the hypotheses that have been postulated so far (for example, monotonicity), but they can be appropriately modified.

a change that improves the technology. Person 2 will offer one unit of numeraire at any price. Thus, the price of output must equal the marginal cost for $y \geq f(1)$, i.e., $p = 2$. But at this price Person 1's demand is zero. Therefore, in the proportional solution Person 2 receives all profits, and the allocation is $(x_1, y_1) = (1, 0)$, $(x_2, y_2) = (0, 10/4)$. The first person's utility is now $U_1(x_1, y_1) = 1 < 5/4$, i.e. her welfare has worsened due to the technological improvement.

Some computations performed by Ignacio Ortuño and presented in Roemer and Silvestre (1987) may clarify the comparison of the three solutions. The values of the price and quantity variables as well as the utility levels are computed there for several two-person economies, with varying preferences, technologies and initial endowments. We were surprised by the similarity between the values corresponding to the proportional solution and the constant-return-equivalent solution. The equal-benefit solution, on the other hand, generated substantially different allocations.

A particularly enlightening sequence of simulations considers eight economies identical except for the technology. The two individuals own the same initial endowments ($\omega_1 = \omega_2 = 1/2$), but their preferences differ. The utility functions are:

$$U_1(x_1, y_1) = 1000\,(x_1)^{0.1}\,(y_1)^{0.9},$$

$$U_2(x_2, y_2) = 1000\,(x_2)^{0.9}\,(y_2)^{0.1},$$

i.e., the first person derives much less utility from leisure than the second. We call them Industrious and Lazy. The technology is given by: $f(L) = (L/2)^a$, for $L/2 < 1$. The parameter a is always between zero and one, but a decreases along the sequence, i.e., the technology improves, because the smaller a, the larger $(L/2)^a$ as long as $L/2$ is between zero and one. We obtain the following results in utility terms:

	Equal-benefit solution		Proportional solution		Constant-return -equiv. solution	
	Industrious	Lazy	Industrious	Lazy	Industrious	Lazy
a	U_1	U_2	U_1	U_2	U_1	U_2
1.0	194	337	194	337	194	337
0.9	211	356	220	342	219	342
0.7	253	401	285	350	284	352
0.6	279	428	326	354	324	357
0.55	294	443	349	356	347	360
0.5	310	459	375	357	371	362
0.1	542	643	709	356	695	388
0.05	608	679	789	351	774	393
0.025	653	699	842	348	825	396

The information is graphically represented in Figure 6.

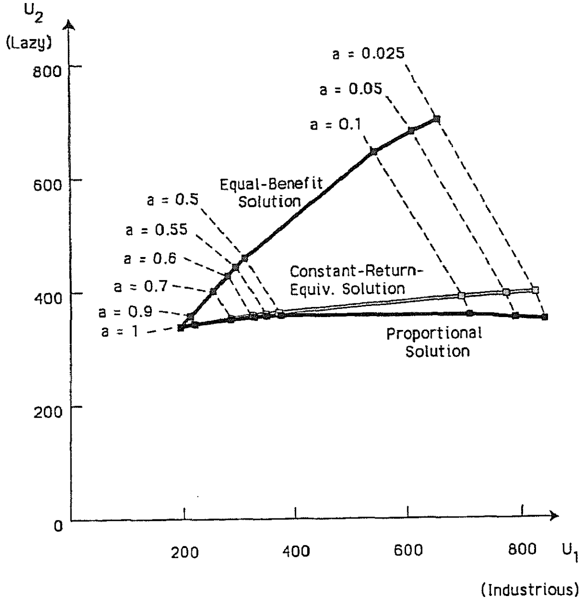

Figure 6
Utility Pairs in the Three Solutions as the Technology Improves

We check that the three solutions coincide when $a = 1$, i.e., when returns to scale are constant. For a close to one, the difference between the solutions is small, and it increases as a decreases.

The numbers reflect the difference between the equal-benefit solution and the other two, a difference that also appears in other simulations not presented here. Technological improvement substantially benefits Lazy in the equal-benefit solution: she receives half the profits essentially generated by the work of Industrious, who spends 80 to 87 percent of her time working. But Lazy receives only a small share of profits in the other two solutions, approximately 10 percent in the proportional solution, and an even lower percentage –which decreases as a increases– in the constant-return-equivalent solution. Technological improvement then translates into an increase in the utility of Industrious, who is the one who actually uses the technology, while Lazy's utility does not change much.

The difference between the computed values for the constant-return-equivalent solution and the proportional solution is not numerically large, but is qualitatively significant. We know that the constant-return-equivalent solution satisfies technological monotonicity. Therefore, as technology improves, we should observe some improvement in Lazy's utility, and that is precisely what our figures show: U_2 increases, perhaps just a little bit, as a decreases. There is no a priori reason, on the contrary, for the proportional solution to be technologically monotonic, and, in fact, it is not in our examples: we can observe that the U_2 corresponding to the proportional solution decreases as the technology improves, for $a < 0.5$.

3.15. Generalizations

The proportional and equal-benefit solutions can be generalized to economies with arbitrary numbers of individuals and goods, inputs or outputs, publicly or privately owned, with intermediate goods and including privately owned firms coexisting with the public sector (see Roemer and Silvestre, 1993). The existence of equal-benefit solutions is, in convex economies, equivalent to the existence of a general competitive equilibrium with an exogenous, egalitarian distribution of profits. Therefore, existence is guaranteed under the usual conditions. But the existence of proportional solutions is not equivalent to the existence of a competitive equilibrium, because the profit shares are not exogenous and they must be determined by the equilibrium conditions. Nevertheless, the technique in the existence proofs by Takashi Negishi (1960) and Kenneth Arrow and Frank Hahn (1971) can be adapted to the proportional solution for convex economies (see Roemer and Silvestre, 1993).

As was the case for the balanced, linear cost-share equilibrium of the Section 2.13, the proportional and equal-benefit solutions do not require profit maximization. Thus, they may in principle exist under increasing returns to scale. Existence is, in particular, guaranteed when consumers are identical, in which case the «natural» solution defined in Section 3.8 is both the proportional and the equal-benefit solution.

The constant-return-equivalent solution, on the contrary, does not generalize to economies with two or more inputs. In fact, there is no continuous mechanism simultaneously satisfying efficiency, technological monotonicity and the condition of selecting the «natural» solution in linear technologies with more than one input. But this impossibility is not pathological because, on reflection, the condition of technological monotonicity is not a natural feature of the public ownership of the technology when there are two or more inputs. Consider the following example. The output is energy, generated via a publicly owned technology by using either oil or straw as fuel. Oil is scarce and its ownership is concentrated in one person, say Person 1. Straw is abundant, but given the present technology, is not very useful as an input. Consider now a technological improvement that greatly increases the productivity of straw while increasing only slightly the productivity of oil. The technical improvement reduces the social value of oil, and it is thus natural to see a decrease in the welfare to the first person. But technological monotonicity would require that the owner of oil benefit from the introduction of the new technology.

Summarizing, technological monotonicity is not a natural connotation of public ownership where there are two or more inputs. This observation, in turn, casts doubt on the requirement of technological monotonicity itself, even for one-input models, because the number of inputs reflects nothing but a convention on the manner in which inputs are aggregated. Moreover, it is not in general true that any improvement in a privately owned resource or technology must necessarily benefit its owner: this was shown by Andreu Mas-Colell a few years back (1976), and also by a variety of papers in the theory of international trade (see Jagdish Bhagwati, 1987) as well as, more recently, in the axiomatic theory of welfare economics. All things considered, it does not seem reasonable to insist on technological monotonicity as a natural property of public ownership.

4. Concluding comments

We have explored different approaches to the question of how the benefits from a publicly owned technology should be distributed. A solution is singled out that attempts to capture the public ownership of the technology together with the private ownership of the input. In the public good case, the solution follows Lindahl's «benefit principle», i.e., the idea that users' contributions towards the cost of supplying the public good should be in line with the individual marginal benefit derived from its use. The solution can also be applied to the case of private goods by reinterpreting the price as a public good or the allocation of output as a vector of private goods. The resulting solution concept is characterized by the fact that a person's consumption of output is proportional to her contribution of input, or, equivalently, that each consumer pays average cost in net terms.

More generally, the present text aims at encouraging the discussion of the conceptual and theoretical questions that public ownership raises. The notion of a «private ownership economy» and the associated equilibrium concept (Gerard Debreu, 1959, Section 5.5) has rightly earned the central place in modern economics by providing a precise analysis of economies where all resources and technologies are privately owned. Considerations both of realism and of social welfare dictate that we move forward and incorporate publicly owned resources and technologies in the analysis. The present text attempts a first step in this direction. Let us hope that future research will achieve substantial progress.

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Resumen

El presente texto, basado en las Lecciones Germán Bernácer de 1990 (Universidad de Alicante), considera la cuestión de cómo deberían asignarse los bienes y recursos de una economía en la que la tecnología es de propiedad pública mientras que el input trabajo es de propiedad privada. Propone una solución según la cual a) si el output es un bien público, entonces las contribuciones de los usuarios siguen el «principio del beneficio» de Lindahl; b) si el output es un bien privado, entonces el consumo de output por parte de una persona es proporcional a su contribución de input o, de forma equivalente, los consumidores pagan, en términos netos, el coste medio.

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