

A TWO-FACTOR DURATION MODEL FOR INTEREST RATE RISK MANAGEMENT*

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This paper estimates a two-factor model of the term structure of interest rates in the Spanish Public Debt market which is compatible with a wide range of term structure movements as those observed during 1994. First, the spot rates to be used as proxies of the unknown factors are chosen, and the resulting model is compared with other competing benchmarks. Then, two further developments are carried out to measure price-risk (through a duration vector) and the impact of interest rate changes on the returns of fixed-income portfolios.

1. Introduction

In an article published in 1990, Elton, Gruber and Michaelly (EGM) pointed out that the term structure of interest rates (TSIR) can be modeled assuming that its behavior can be described by a small number of state variables. Equilibrium fixed income valuation models assume that bond prices are a function of a small number of variables that follow a diffusion process¹. The proxies of the state variables are usually spot rates of different maturities. However, empirical testing of those models has not shown their superiority against other simpler ones. EGM argue that this result can be caused by the fact that the spot interest rates chosen for modeling the term structure have been selected arbitrarily. So, they propose to analyze empirically the TSIR to determine which spot rates can best be used to model the whole behavior of the TSIR.

In this article we present a two-factor model of the TSIR, based on the methodology developed by EGM, which is applied to the Spanish Public Debt

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¹ Among those models we include one-factor models (such as Cox, Ingersol and Ross, 1985, Brennan and Schwartz, 1977, and Vasicek, 1977) as well as two-factor models (such as Cox, Ingersol and Ross, 1985, Richard, 1978, Brennan and Schwartz, 1979, 1980, and Nelson and Schaefer, 1983).

market. The paper is organized as follows. In Section 2, we briefly explain the EGM methodology to find the best spot interest rates to be used as proxies of the unknown factors and their closed model of the TSIR. Section 3 describes the data. Section 4 shows the results obtained: the optimal proxies for the state variables in both one-factor and two-factor models (Subsection 4.1), a closed TSIR model (Subsection 4.2) and finally an evaluation of different competing models using out of sample data (Subsection 4.3). It is shown that the EGM methodology leads to select proxies that clearly outperform the proxies used elsewhere. In Section 5, two extensions of these models are developed. First, a duration vector is defined to measure price risk against a wide range of term structure shifts. Second, a model to analyze the impact of interest rate changes on the returns of fixed income portfolios is developed and tested using data from the Spanish Public Debt market.

2. Methodology

EGM assume that unexpected changes in spot interest rates are linearly related to two unknown factors, F_1 and F_2 as follows:

$$dr_{i,t} = \beta_{i,0} + \beta_{i,1} dF_{1,t} + \beta_{i,2} dF_{2,t} + \varepsilon_{i,t}, \quad [1]$$

where $dr_{i,t}$ is the unexpected change during the time interval $[t - 1, t]$ of the i -period spot interest rate, $dF_{1,t}$ and $dF_{2,t}$ denote unexpected changes in the unknown factors during period $[t - 1, t]$, and $\varepsilon_{i,t}$ is an error term.

They suggest the use of two different spot interest rates as proxies of the unknown factors; additionally they require the mean of unexpected changes of interest rates to be equal to zero so $\beta_{i,0}$ should be zero. Thus, the model they propose becomes:

$$dr_{i,t} = a_i dr_{x,t} + b_i (dr_{z,t} - dr_{x,t}) + \varepsilon_{i,t}, \quad [2]$$

where $dr_{i,t}$ is defined as before, $dr_{x,t}$ and $dr_{z,t}$ denote unexpected changes in the optimal rates to be used as proxies of unknown factors², and $\varepsilon_{i,t}$ is a normally distributed random term with zero mean and constant variance σ^2 .

As a particular case of this model, we have one-factor models where unexpected changes of interest rates are assumed to depend linearly on a unique optimal interest rate used as a proxy of the unknown factor.

The determination coefficient between r_i and the independent variables r_x and $(r_z - r_x)$ is given by

$$R_{i,(x,z)}^2 = 1 - \frac{Var(\varepsilon_{i,t})}{Var(dr_{i,t})}, \quad [3]$$

or equivalently,

$$R_{i,(x,z)}^2 \cdot Var(dr_{i,t}) = Var(dr_{i,t}) - Var(\varepsilon_{i,t}). \quad [4]$$

² In fact, the second proxy is the difference between dr_z and dr_x instead of dr_z in order to avoid multicollinearity problems.

According to this equation, to minimize the residual term variance is equivalent to maximizing the left hand side of this equation

$$R_{i,(x,z)}^2 \cdot \text{Var}(dr_i). \quad [5]$$

Under the former hypothesis, the proxies of the unknown factors can be ranked by the variance of the spot rate changes times the regression coefficient. This technique allows to determine the proxies that best explain unexpected changes of the spot rate with a given maturity.

The problem now is to find out the best proxies for describing a set of interest rates, i.e., the term structure³. The procedure that EGM suggest consist of using an interest rate weighting scheme. In particular, they obtain the optimal spot rates by maximizing the following objective function:

$$\text{Max}_{x,z} \sum_i \omega_i \cdot R_{i,(x,z)}^2 \cdot \text{Var}(dr_i), \quad [6]$$

where ω_i is the weight assigned to the i -period spot interest rate and represents the importance of each interest rate being explained by the chosen proxies of the unknown factors.

It is important to note the role played by the weights. For instance, if we want to study the effects of term structure shifts on a short term fixed-income portfolio we would be interested in a model that explains, above all, short term interest rates, so we should assign a high value to the weights corresponding to short term interest rates. In this case, we would not mind using a model with a poor explanatory power for long term interest rates. This is very important if we take into account that the model can be used for evaluating interest rate risk management. So, if we are going to analyze the management of a standard Spanish bond portfolio we should have a model that describes, as well as possible, the interest rates corresponding to the maturity of most of the bonds that are included in such a portfolio. So ω_i should be chosen accordingly.

An additional problem that arises when applying the EGM methodology is related to the availability of TSIR estimations. If spot interest rates (or zero coupon bonds rates) are not directly observable it is necessary to estimate the TSIR. In this paper, Spanish term structure estimations were obtained applying a modified version of Vasicek and Fong (1982) model, who suggest the use of exponential splines^{4,5}. In particular, we used exponential splines

³ This set consists of 28 spot interest rates. They are the spot rates from one to eighteen months (month by month), the twenty-one month rate and the rates from two years to ten years (year by year).

⁴ For a more detailed description of the model see Navarro, Ferrer and Nave (1995) and Contreras, Ferrer, Navarro and Nave (1996).

⁵ Term structure estimation has become one of the main topics in the field of interest rate risk management. In Spain, there are some recent papers which approach this problem using different methodologies; see Ezquiaga (1994), Lamothe, Leber and Soler (1995), and Núñez (1996). For a exhaustive description of term structure estimation methods see Anderson *et al.* (1996). As these authors point out «a major factor affecting the choice of which model to use is the purpose to which it is to be put. Thus for macroeconomic analysis, a parsimonious model ... may be deemed appropriate, while at the other end of the spectrum the highly flexible non-parametric approaches may be better suited for pricing (through some practitioners also use parsimonious models for pricing)».

with a unique variable knot. The parameters of the model and the knot position were estimated by applying generalized least squares due to the heteroskedasticity of the error term⁶.

Once TSIR are estimated, the next step is to define what is meant by unexpected interest rate changes⁷. Like EGM, we consider two alternatives. The first one (case 1) assumes that the Pure Expectations Theory holds exactly so, unexpected changes of interest rates during a period $[t - 1, t]$ will be defined as the difference between the spot rates at t minus the corresponding forward rate on the basis of the term structure observed at $t - 1$. The second (case 2) considers all changes of interest rates to be unexpected, which is consistent with the Market Segmentation Theory about the term structure of interest rates.

Once optimal rates (r_x and r_z) are determined by solving program [6], unexpected changes of i -period interest rates can be modeled, according to EGM, as follows:

$$\Delta R_{i,t} = \hat{a}_i \Delta R_{x,t} + \hat{b}_i \Delta (R_{z,t} - R_{x,t}) + \varepsilon_{i,t}, \quad [7]$$

where ΔR_x and ΔR_z represent unexpected changes in the two optimal interest rates used as proxies of the unknown factors.

Alternatively, the model can be rewritten as:

$$\Delta R_{i,t} = (\hat{a}_i - \hat{b}_i \cdot \Delta R_{x,t} + \hat{b}_i \Delta R_{z,t} + \varepsilon_{i,t}, \quad [8]$$

where unexpected changes of interest rates can be expressed directly from unexpected changes in R_x and R_z .

The values of $(\hat{a}_i - \hat{b}_i)$ and \hat{b}_i which are estimated by OLS can be interpreted as the sensitiveness of the i -period interest rate to changes in the two optimal rates r_x and r_z , respectively.

Finally, to obtain a closed term structure model EGM suggest to interpolate the $(\hat{a}_i - \hat{b}_i)$ and \hat{b}_i values in order to capture the sensitiveness of any interest rate—not only the spot rates initially selected—to changes in the two key spot rates. In particular, they suggest the following model:

$$a_i = c_0 + c_1 \ln(i) + c_2 (\ln(i))^2 + \varepsilon_i, \quad [9]$$

pointing out that its R^2 is greater than 97% in the one-factor model and greater than 93% and 99% for the two coefficients of the two-factor model.

3. Data

The data used in this article were term structure estimations corresponding to the last trading day of each week from 8 January 1993 to 30 June 1996

⁶ The variance of the error term is assumed to depend on bond Macaulay's duration and the interest rate volatility term structure. See Ferrer, Navarro and Nave (1995) for a more detailed description of the model hypothesis.

⁷ For a review of the different hypothesis about the TSIR see, for example, Freixas (1992).

(see Figure 1). This TSIR estimations were obtained applying Vasicek and Fong (1982) methodology to the mean daily prices of all the assets traded in the Spanish Public Debt market⁸.

In order to test the models presented in this paper, the whole sample period (January 1993 to June 1996) was divided into two sub-periods. The first one is the period from 8 of January 1993 to 1 of December 1994 (we refer to it as period 1); model parameters were estimated using these data. The second part of the sample covers the period from 1 December 1994 to 30 June 1996 (we refer to it as period 2); these data were used to perform out of sample tests.

An additional problem we have to deal with was the existence of a possible structural change in the behavior of interest rates in the Summer of 1993 when the widening of the fluctuation bands of the European Monetary System (EMS) took place and monetary authorities gave up using interest rates as an instrument to defend the Spanish currency. It is easy to see the change in short term interest rate volatility (see Figure 2). So, we subdivided the first sample period into two sub-samples. The first one includes unexpected weekly changes of interest rates from January 1993 to August 1993 (period 1.1) and the second from September 1993 to November 1994 (period 1.2).

Finally, we have to specify the weighting scheme (ω_i values) for the EGM model. These authors suggest two alternatives. The first one gives the same importance to each spot interest rate, so that all ω_i are equal⁹. The second scheme assigns to each spot rate the relative importance that they have when valuing a standard fixed income portfolio. Since the aim of this paper is to develop a model for analyzing the effect of interest rate changes on the market value of Spanish fixed income portfolios, we studied the cash flows generated by a «standard» Fondtesoro-FIM¹⁰. Specifically, we analyzed the cash flows generated by the five most important Fondtesoros-FIM at 30 September 1994¹¹. These cash flows were classified into 28 groups according to the date these cash flows are due. Afterwards, cash flows were discounted at the midpoint spot interest rate of the interval they belong to, using the term structure estimated at 30 September 1994. The ω_i value assigned to

⁸ See «Boletín de la Central de Anotaciones en Cuenta» del Banco de España.

⁹ Specifically, equal weights were given to each yearly interval so if, for example, the first year interval contains information about 12 different spot rates, each spot rate receives only 1/12 of the yearly weight. In a previous paper, Contreras, Ferrer, Navarro and Nave (1994) tested the robustness of EGM's results through the application of the same weighting scheme to the Spanish market. In fact, similar results are obtained despite the differences which exist in both papers: different countries, different periods and different periodicity in interest rates changes.

¹⁰ A Fondtesoro-FIM is a mutual fund that invests mainly in long term Public Debt.

¹¹ According to information provided by supervisory and control authorities, the five Fondtesoro FIMs with the highest net worth are: Ahorro Corporación Deuda (130.916 M.), BBV Deuda (51.112 M.), Invermadrid (37.540 M.), Fondeuda Ahorro (34.481 M.) and A. B. (25.752 M.). Unfortunately, the repo-depo market assets have not been included in the analysis because the data referring to their maturity were not available.

each interest rate was the percentage of the total present value of the whole portfolio that is generated by the cash flows due at each interval.

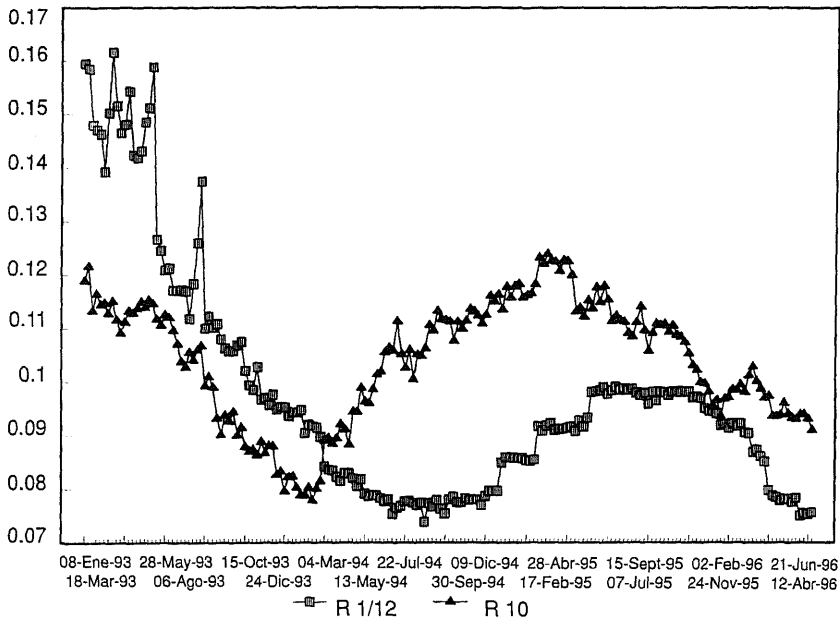


FIGURE 1

One-month and ten-year spot interest rates

Spot interest rates are estimated using a modified version of Vasicek and Fong (1982) model. The sample period goes from January 1993 to June 1996.

4. Empirical results

4.1. Optimal proxies of the unknown factors

In this section we apply the EGM methodology to obtain the optimal rates to be used for describing the behavior of the Spanish TSIR using weekly data corresponding to Period 1 (January 1993 to November 1994).

In the case of one-factor models (Table 1), this methodology led to the use of the three-year spot rate as the best proxy of the unknown factor instead of the rates used in alternative one-factor models¹². The changing behavior of interest rates during period 1 suggested the convenience of dividing the sample into two sub-samples: the period prior to the EMS crisis and the period after the widening of the fluctuation bands of the EMS (periods 1.1. and 1.2 respectively). However, according to Table 1 the three-year interest

¹² Nelson and Shaefer (1983) used the 13-year spot rate and Babbel (1983) the one-year spot rate. Instead of 13-year interest rate we have compared our optimal rate with the ten-year spot rate which is the longest maturity interest rate available during the whole sample period.

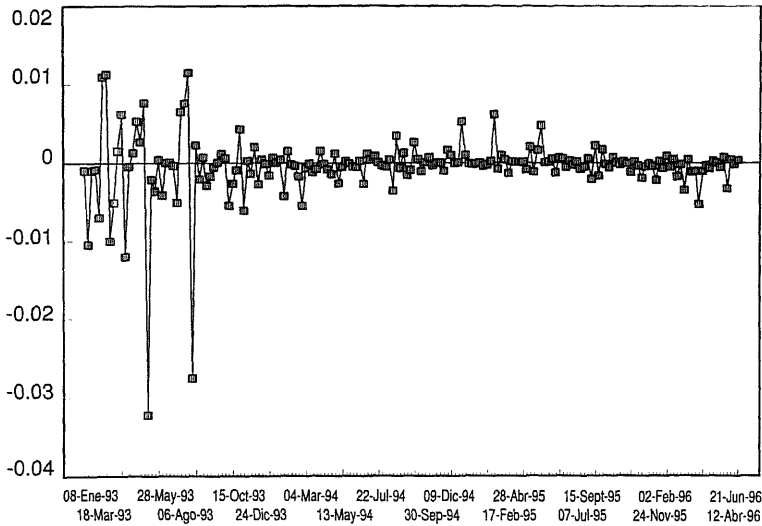


FIGURE 2
One-month interest rate weekly changes

TABLE 1
Optimal spot rates in one-factor models

Entries in the table are values of the objective function [6] corresponding to different proxies of the unknown factor in one-state variable models. Periods 1, 1.1, and 1.2 cover from January 1993 to November 1994, from January 1993 to August 1993, and from September 1993 to November 1994, respectively

Case 1 (Pure Expectations Theory)				
	Optimal rate ¹	3 year ²	1 year ³	10 year ⁴
Period 1	5.90E-05 (3 yr)	5.90E-05	4.04E-05	3.96E-05
Period 1.1	6.70E-05 (2 yr)	6.66E-05	4.93E-05	3.28E-05
Period 1.2	5.22E-05 (3 yr)	5.22E-05	3.70E-05	4.09E-05
Case 2 (Market Segmentation Theory)				
	Optimal rate ¹	3 year ²	1 year ³	10 year ⁴
Period 1	5.87E-05 (3 yr)	5.87E-05	4.20E-05	3.92E-05
Period 1.1	7.19E-05 (2 yr)	7.06E-05	5.58E-05	3.23E-05
Period 1.2	5.24E-05 (3 yr)	5.24E-05	3.66E-05	4.09E-05

¹ The number in parenthesis is the maturity with the maximum value.

² Optimal proxy according to the EGM methodology.

³ Spot rate used to model the TSIR in Babbel (1983).

⁴ Spot rate used to model the TSIR in Nelson and Shaefer (1983).

rate outperformed the other two competing proxies. In fact, the three-year spot rate was the best or the second best among the whole set of interest rates used to describe the term structure in both sub-periods.

In the case of two-factor models (Table 2), the key rates chosen as proxies of the factors were the three-year and the two-month spot rates. These are the optimal rates for the whole period 1, but they performed very well in both sub-periods. The objective function values were very close to the optimum and, in any case, they outperformed the other two alternative benchmarks^{13, 14}.

TABLE 2
Optimal spot rates in two-factors models

Entries in the table are values of the objective function [6] corresponding to different proxies of the unknown factors in two-state variable models. Periods 1, 1.1. and 1.2 cover from January 1993 to November 1994, from January 1993 to August 1993, and from September 1993 to November 1994, respectively

Case 1 (Pure Expectations Theory)				
	Optimal rate ¹	3 yr & 2 m ²	6 yr & 8 m ³	10 yr & 5 yr ⁴
Period 1	7.56E-05 (3 yr; 2 m)	7.56E-05	6.12E-05	5.47E-05
Period 1.1	7.40E-05 (3 yr; 2 m)	7.40E-05	6.25E-05	5.60E-05
Period 1.2	5.60E-05 (2 yr; 5 yr)	5.53E-05	4.76E-05	4.85E-05
Case 2 (Market Segmentation Theory)				
	Optimal rate ¹	3 yr % 2 m ²	6 yr & 8 m ³	10 yr & 5 yr ⁴
Period 1	7.84E-05 (3 yr; 2 m)	7.84E-05	6.55E-05	5.70E-05
Period 1.1	11.01E-05 (2 yr; 2 m)	10.94E-05	9.05E-05	7.29E-05
Period 1.2	5.61E-05 (2 yr; 5 yr)	5.57E-05	4.80E-05	4.85E-05

¹ The number in parenthesis are the maturities with the maximum value.

² Optimal proxies according to the EGM methodology applied to Spanish data.

³ Optimal proxies according to the EGM methodology applied to U.S. data.

⁴ Spot rates used to model the TSIR in Nelson and Shaefer (1983).

¹³ Nelson and Shaeffer (1983) used the 13-year and the 5-year spot rate to describe the term structure. When comparing with our optimal rates we substitute the 13-year rate by the 10-year for the reasons argued in footnote 15. Six-year and eight-month were the optimal interest rates used by EGM.

¹⁴ Had we used an equally weighting scheme (as EGM suggest) the optimal rates would have been the 7-year spot rate in one-factor model and the 7-year and 3-month interest rates in two-factor models. But this weighting scheme, which is appropriate to analyze ladder portfolios, would lead to a poor description of the term structure around 2-3 years maturities. However, these are the most important interest rates to be described, as most of the cash flows generated by Spanish mutual funds are concentrated around that term. In any case, the stability of the proxies corresponding to this alternative weighting scheme was studied leading to similar results.

It is worth pointing out that both cases (Pure Expectations Theory and Market Segmentation Theory) led to the same results. This is due to the fact of using weekly interest rate changes. In fact, when short time intervals are considered (for instance, daily or weekly changes), only if the term structure shows an extremely pronounced slope significant differences between case 1 and case 2 may appear.

The $R_{\hat{i}; (2/12,3)}$ values (that provides the percentage of the variance of the i -period spot interest rate which is explained by the model) range from 100% for two-month and three-year interest rates to a lowest level of 52% corresponding to the 9-month rate. This result suggests the convenience of including a third factor to explain term structure behavior around one-year maturity. As Strickland (1993) suggests, this third factor would probably account for curvature shifts of the term structure of interest rates.

Another interesting result is the fact that $(\hat{a}_i - \hat{b}_i)$ values (which indicate the response of interest rates to changes in the two-month interest rate) are not significantly different from zero for terms greater than three years and, in a similar way, the $\hat{b}_{1/12}$ value (which shows the responsiveness of interest rates to changes in the three-year interest rate) is very close to zero. Thus, weekly changes of spot rates with a maturity greater than three years are independent of the behavior of the two-month interest rate. In the same way, weekly changes of interest rates with a maturity less than two months are independent of changes in the other key spot rate, i.e., the three-year interest rate.

4.2. *A closed term structure model*

In order to develop a closed TSIR model, EGM suggest to interpolate the \hat{b}_i and $\hat{a}_i - \hat{b}_i$ values to obtain the sensitivity of all interest rates to changes in the two key rates $R_{2/12}$ and R_3 .

To address this problem, we tried a wide family of functions and finally we selected two rational functions. In particular, to fit the $\hat{a}_i - \hat{b}_i$ values –coefficients that measure the sensitivity of interest rates to changes in the two-month interest rate– the function chosen was ¹⁵:

$$(a - b)(i) = \frac{k_0 + k_1 \ln(i) + k_2 \ln(i)^2}{1 + k_3 \ln(i) + k_4 \ln(i)^2}; \quad k_j \in R \quad [10]$$

With respect to the coefficients \hat{b}_i , the selected function was:

$$b(i) = \frac{h_0 + h_1 \ln(i) + h_2 \ln(i)^2}{1 + h_3 \ln(i) + h_4 \ln(i)^2 + h_5 \ln(i)^3}; \quad k_j \in R \quad [11]$$

¹⁵ We use notation $a_i - b_i$ and $(a - b)(i)$ indistinctly as well as b_i and $b(i)$.

However, the sensitivity functions should be constrained by making $\hat{a}_{2/12} - \hat{b}_{2/12}$ and \hat{a}_3 equal to one. For the same reason, $\hat{a}_3 - \hat{b}_3$ and $\hat{b}_{2/12}$ should be equal to zero. So, these two functions were constrained as follows:

$$\begin{aligned} (1) (a - b)(2/12) &= 1; & (2) (a - b)(3) &= 0; & (3) (a - b)'(3) &= 0 \\ (4) b(2/12) &= 0; & (5) b'(2/12) &= 0; & (6) b(3) &= 1 \end{aligned} \quad [12]$$

Constraints (3) and (5) in [12] are imposed in order to guarantee the smoothness of the function used to describe the responsiveness of interest rates to changes in the two proxies of the unknown factors. It must be taken into account that these constraints reduce the number of parameters to be estimated to two for $(a - b)(i)$ and to three for $b(i)$. The results are depicted in Figures 3 and 4. The R^2 values for the curves fitted by OLS were 99.89% for $(a - b)(i)$ and 99.94% for $b(i)$ in case 1, and 99.78% for $(a - b)(i)$ and 99.89% for $b(i)$ in case 2. These outcomes clearly outperform those obtained by EGM (see [9]). This may explain some of the results obtained in the next section when comparing different competing models.

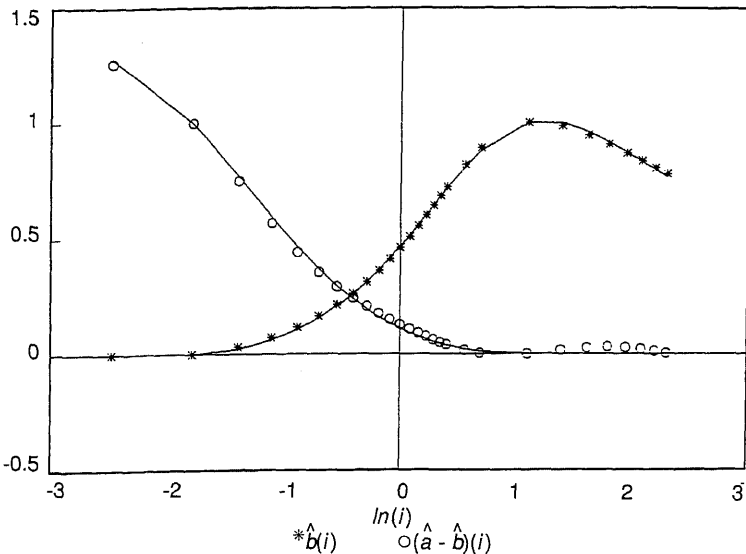


FIGURE 3

Sensitivity estimates of spot interest rates to changes in optimal proxies under the pure expectations hypothesis (case 1)

$\hat{b}(i)$ measures the responsiveness of i -period spot rates to changes in the three-year spot rate.

$(\hat{a} - \hat{b})(i)$ measures the responsiveness of i -period spot rates to changes in the two-month spot rate.

– adjusted $b(i)$ and $(a - b)(i)$ values according to [10] and [11], respectively.

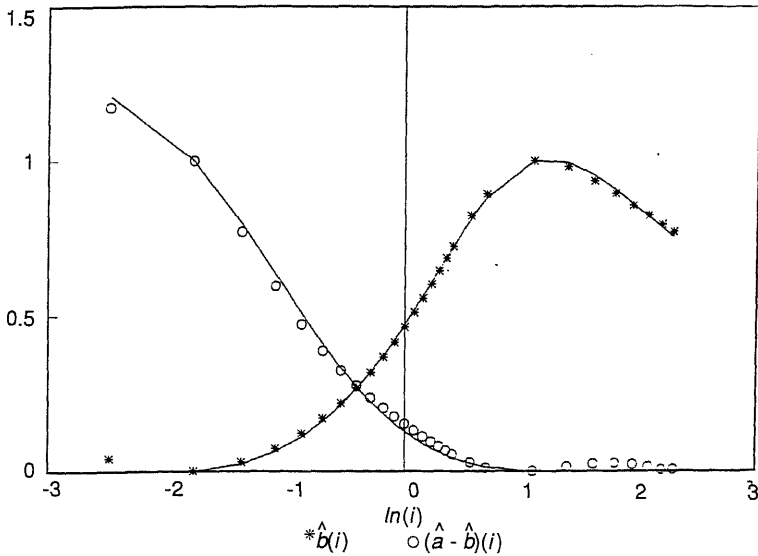


FIGURE 4

Sensitivity estimates of spot interest rates to changes in optimal proxies under the market segmentation hypothesis (case 2)

$\hat{b}(i)$ measures the responsiveness of i -period spot rates to changes in the three-year spot rate.

$(\hat{a} - \hat{b})(i)$ measures the responsiveness of i -period spot rates to changes in the two-month spot rate.

– adjusted $b(i)$ and $(a - b)(i)$ values according to [10] and [11], respectively.

4.3. Out of sample comparison of alternative models

In order to compare and rank different models of the term structure, we carried out some tests using out of sample data (weekly changes of interest rates corresponding to period 2, i.e. from December 1994 to June 1996).

In particular, we computed the mean squared error of unexpected interest rate changes for each maturity, so, for each model, we had 28 data points corresponding to the mean squared error from one month to ten year interest rates. Then, as EGM suggest, we calculated the total mean squared error and the standard deviation using the weighting scheme discussed earlier. Assuming that the mean squared error for each maturity is normally distributed (using a central limit theorem) we estimated confidence intervals in order to analyze the significance of the difference of the whole mean squared error between alternative models. The results obtained are shown in Table 3.

As we can observe, the two-factor optimal model (three-year and two-month rates) outperformed all other models during period 2. The main difference with the EGM results is that the closed two factor model is in both cases (1

and 2) the second best with a difference that is not significant at the 90% level with respect to the optimal model. This is due, probably, to the different way we adjusted the factor sensitivities formulae $b(i)$ and $(a - b)(i)$.

TABLE 3
Ranking of the competing models using out of sample data¹

Case 1 (Pure Expectations Theory)		Case 2 (Market Segmentation Theory)	
Factor proxies	Mean square error	Factor proxies	Mean square error
3 yrs & 2m	3.58E-07	3 yrs & 2 m	3.60E-07
3 yrs & 2 m (closed model)	3.62E-07	3 yrs & 2 m (closed model)	3.72E-07
3 yrs	5.56E-07*	3 yrs	6.00E-07**
6 yrs & 8 m	7.79E-07*	6 yrs & 8 m	7.90E-07*
Naive (3 yrs)	9.10E-07*	Naive (3 yrs)	9.11E-07*
1 yr	1.36E-06*	1 yr	1.43E-06*
10 yrs	1.82E-06*	10 yrs	1.86E-06*

** Significant difference at 90% level.

* Significant difference at 95% level.

¹ Out of sample data covers the period from December 1994 to June 1996.

An interesting point is that the optimal one-factor model produced better results than those using Macaulay duration (Naive) or any other benchmark, including the two-factor models used by other authors.

Another result that should be highlighted is that the Macaulay measure beat all the other one-factor models. This can explain why Macaulay duration, as an interest rate risk measure, is really difficult to improve (in fact the difference with the one factor optimal model is not significant at the 95% level).

In summary, as EGM point out, «this technique for choosing the optimal proxies» of the unknown factors of the term structure, «worked well with both one and two variables model», and «overall, the two-variables model outperforms the one-variable model, and all outperformed benchmark models».

5. Duration analysis

The use of effective duration as a measure of interest rate risk exposure has been criticized as it assumes implicitly parallel shifts of whole the term structure (Bierwag, 1987, and Douglas, 1990). In fact, term structure movements can be classified in three main groups: parallel shifts, changes in slope and changes in curvature. In the Spanish Public Debt market, level changes only accounted for around a 70% of weekly term structure changes during 1993-1995 (Navarro and Nave, 1996)¹⁶. Moreover, if we observe the

¹⁶ This result is very similar to those obtained for other countries. See McLeod (1990), Steeley (1990), Litterman and Scheinkman (1991), Strickland (1993), and D'Ecclesia and Zenios (1994).

behavior of the term structure during the last years (see Figure 1), we can see how short term interest rates followed a decreasing path while long term interest rates increased around 300 basis points since January 1994 to December 1994.

This sort of facts has led to different duration measures each of them depending on different assumptions about term structure movements. Simonson and Stock (1991) tested the effectiveness of these alternative one factor duration measures concluding their relative failure to predict security value changes under regimes of unstable term structures. Other authors (Chambers, Carleton and McEnally, 1988, Reitano, 1990, and Ho, 1992) suggest the concept of duration vector to measure the sensitivity of bonds and other more complex instruments against interest rate changes.

In this context, this section presents two extensions of the closed TSIR model developed earlier. First, a duration vector is obtained to measure the impact of unexpected changes in the two optimal interest rates on the value of fixed-income portfolios. Then the model is developed to analyze the effect of term structure changes on bond portfolio returns which can be useful for measuring bond portfolio performance. Finally, this model is tested using data from the Spanish Public Debt market.

5.1. Duration vector as a price risk measure

According to the closed TSIR model developed in Section 4, interest rate changes can be described approximately by the following equation:

$$\Delta R_{i,t} = (a - b)(i)\Delta R_{2/12,t} + b(i)\Delta R_{3,t} + \varepsilon_{i,t} \quad [13]$$

where $(a - b)(i)$ is the sensitivity coefficient of the i -period spot rate to changes in the two months spot interest rate, $b(i)$ is the sensitivity coefficient of the i -period spot rate to changes in the three years spot interest rate, and $\varepsilon_{i,t}$ is an error term with $E[\varepsilon_{i,t}] = 0$ and $\text{VAR}[\varepsilon_{i,t}] = \sigma^2$.

In addition, we assume¹⁷ that bond prices can be expressed as:

$$P^k = \sum_{\tau=1}^{\rho} C^k(1 + R_{t,\tau})^{-t_{\tau}^k} + N^k(1 + R_{t,\rho})^{-t_{\rho}^k} + \gamma \cdot W^k + \varepsilon^k, \quad [14]$$

where ρ is the number of coupon payments generated by k -th bond, t_{τ}^k are the time coupon payments are due, t_{ρ}^k is the date the bond principal is repaid, C^k is the amount of coupon payment, N^k is the bond principal, γW^k is a tax premium that depends on the coupon features of the k -th bond.

¹⁷ This is the bond valuation model used by Vasicek and Fong to develop their TSIR estimation methodology.

Variable W^k is defined to capture the impact of taxation on the bond price, and it can be approximated by the following formula¹⁸:

$$W^k \approx \sum_{r=1}^p C^k (1 + R_{t^r})^{-t^r}. \quad [15]$$

Then, the bond price is given by:

$$P^k \approx \sum_{r=1}^p C^k (1 + \gamma) \cdot (1 + R_{t^r})^{-t^r} + N^k (1 + R_{t^p})^{-t^p} + \varepsilon^k. \quad [16]$$

From this bond valuation model the change in bond value caused by an unexpected change in the two key spot rates can be estimated by:

$$dP^k = \sum_{r=1}^p \left[\frac{\partial P^k}{\partial R_{t^r}} \cdot \frac{\partial R_{t^r}}{\partial R_{2/12}} \cdot dR_{2/12} + \frac{\partial P^k}{\partial R_{t^r}} \cdot \frac{\partial R_{t^r}}{\partial R_3} \cdot dR_3 \right]. \quad [17]$$

Operating on the right hand side of expression [17] and dividing by P^k , we get:

$$\frac{\Delta P^k}{P^k} \approx -D_1 \Delta R_{2/12} - D_2 \Delta R_3, \quad [18]$$

where

$$D_1 = \frac{\sum_{r=1}^p t_r^k (a-b)(t_r^k) C^k (1+\gamma)(1+R_{t^r})^{-t_r^k-1} + t_p^k (a-b)(t_p^k) N^k (1+R_{t^p})^{-t_p^k-1}}{P^k}, \quad [19]$$

and

$$D_2 = \frac{\sum_{r=1}^p t_r^k b(t_r^k) C^k (1+\gamma)(1+R_{t^r})^{-t_r^k-1} + t_p^k b(t_p^k) N^k (1+R_{t^p})^{-t_p^k-1}}{P^k} \quad [20]$$

Note that if the term structure is flat and the term structure movements are parallel, then D_1 and D_2 would equal Macaulay's modified duration.

These results can be easily generalized to a portfolio of bonds with different maturities. In fact, the relative price change caused by an interest rate movement can be estimated using a portfolio duration calculated as a weighted average of the duration of the bonds included in this portfolio, where the weights are the percentage of the total portfolio value assigned to each bond, i.e.,

¹⁸ See Ferrer, Navarro and Nave (1995).

$$\begin{aligned} \frac{\Delta V}{V} &= \frac{K^1 \Delta P^1 + K^2 \Delta P^2 + \dots + K^m \Delta P^m}{K^1 P^1 + K^2 P^2 + \dots + K^m P^m} \approx \\ &\approx \frac{K^1 [-D_1^1 \Delta R_{2/12} - D_2^1 \Delta R_3] P^1 + \dots + K^m [-D_1^m \Delta R_{2/12} - D_2^m \Delta R_3] P^m}{V} = \quad [21] \\ &= -(\lambda^1 D_1^1 + \lambda^2 D_1^2 + \dots + \lambda^m D_1^m) \cdot \Delta R_{2/12} - \\ &(\lambda^1 D_2^1 + \lambda^2 D_2^2 + \dots + \lambda^m D_2^m) \Delta R_3 = -D_1^c \Delta R_{2/12} - D_2^c \Delta R_3, \end{aligned}$$

where V is the present value of the portfolio, P^k is the k -bond price, K^k is the number of type k bonds in the portfolio, D_1^k y D_2^k are the k -bond durations corresponding to $\Delta R_{2/12}$ and ΔR_3 respectively, $\lambda^k = (N^k \cdot P^k)/V$ is the weight that the k -bond has in total portfolio value, and D_1^c y D_2^c are portfolio durations corresponding to $R_{2/12}$ and R_3 respectively.

5.2. Interest rate risk management

If we assume the pure expectations theory, the expected change in portfolio value during a given period of time (which we call horizon planning period), is:

$$MF = V[(1 + R_H)^H - 1], \quad [22]$$

where MF is the expected change in portfolio value, R_H is the spot rate corresponding to a term equal to the horizon planning period, and H is the length of this period.

Assuming this hypothesis, we are going to quantify the effects of a given unexpected interest rates change on MF. We have to take into account that portfolio value depends on the TSIR and the fact that the term structure depends, according to our model, on two factors, $R_{2/12}$ and R_3 . In this way, we can rewrite the former equation as follows:

$$MF = V(\bar{R})(1 + R_H)^H - V, \quad [23]$$

where $V(\bar{R})$ denotes portfolio present value as a function of the vector of current spot interest rates, \bar{R} is a vector representing the current term structure of interest rates which is assumed to depend on $R_{2/12}$ and R_3 , R_H is the spot interest rate with maturity at the end of the horizon planning period, and V is an escalar which represents the current portfolio value¹⁹.

Now, it is easy to obtain the impact on the expected portfolio return caused by an unexpected change of the two key interest rates ($R_{2/12}$ and R_3)

¹⁹ The main difference between $V(\bar{R})$ and V is that the former is a function depending upon \bar{R} , and so depending on $R_{2/12}$ and R_3 , whereas V is considered as a escalar. I the term structure doesn't change $V(\bar{R}) = V$ but if a term structure movement occurs then $V(\bar{R} + \Delta \bar{R}) \neq V$ denoting by $\bar{R} + \Delta \bar{R}$ the new TSIR. Of course, the value of $\Delta \bar{R}$ depends on $\Delta R_{2/12}$ and ΔR_3 , according to [12].

assuming that the value of the coefficient corresponding to the dummy variable γ remains unchanged. If ΔMF is approximated through the total differential of MF as a function of $R_{2/12}$ and R_3 we have:

$$dMF = \frac{\partial MF}{\partial R_{2/12}} dR_{2/12} + \frac{\partial MF}{\partial R_3} dR_3. \quad [24]$$

Taking into account that

$$\frac{\partial V(\bar{R})}{\partial R_{2/12}} = -V(\bar{R}) \cdot D_1; \quad \frac{\partial V(\bar{R})}{\partial R_3} = -V(\bar{R}) \cdot D_2; \quad \frac{\partial R_H}{\partial R_{2/12}} = (a - b)(H); \quad \frac{\partial R_H}{\partial R_3} = b(H) \quad [25]$$

we get

$$\begin{aligned} \Delta MF \approx V(\bar{R})(1 + R_H)^H \cdot \left\{ \left[\frac{H \cdot (a - b)(H)}{1 + R_H} - D_1 \right] \cdot \Delta R_{2/12} \right. \\ \left. + \left[\frac{H \cdot b(H)}{(1 + R_H)} - D_2 \right] \cdot \Delta R_3 \right\}, \quad [26] \end{aligned}$$

where H is the horizon planning period, D_1 and D_2 are portfolio durations, and $(a - b)(H)$ and $b(H)$ are the sensitivity of spot rates with maturity at the end of the planning period against ΔR_3 and $\Delta R_{2/12}$, respectively.

In order to test the expression [26], we have analyzed the observed returns of a set of equally weighted portfolios randomly chosen from bonds traded in the Spanish Public Debt market²⁰. The horizon planning period we have considered was a quarter, and the whole period analyzed was the last two quarters of 1993, the four quarters of 1994 and 1995 and the first two quarters of 1996²¹. The total number of portfolios and so the total number of data available was 35: three portfolios for each quarter (except the third quarter of 1993 when only two portfolios were considered). Each portfolio consists, at least, of six different bonds in order to reduce the effects of idiosyncratic price fluctuations²².

Portfolio durations were computed for each asset at the beginning of every quarter according to formulae [19] and [20] and TSIR estimations described in Section 2. Portfolio durations were calculated as the arithmetic mean of the durations of the bonds that compose each portfolio as they were equally weighted.

²⁰ The exact assets included in each portfolio can be requested from the authors.

²¹ The exact initial dates of these periods are the first Friday of each quarter and the final date is the first Friday of the next quarter.

²² This is the reason why the first two quarters of 1993 were not included in this analysis. Also, in order to increase the number of data, when a bond outstanding at the beginning of each period didn't trade, we use its theoretical price and durations derived from the term structure estimations described in Section 2.

We would like to point out the fact that D_1 portfolio values range between 1.70 and 3.34; it means that the mean term to maturity of the bonds in each portfolio is between two and four years approximately. Although the bonds of each portfolio were chosen randomly this result matches the weighting scheme proposed in Section 3. This outcome is not surprising and it is derived from the fact that the main Fondtesoros-FIM, on average, have a portfolio composition close to the maturity structure of the whole outstanding Public Debt. If future Treasury bond issues have a maturity structure different from the current one, a new weighting scheme should be applied in order to find out the best proxies to describe the Spanish TSIR.

Finally, we computed the impact of unexpected interest rate changes on portfolio returns. Actual returns were obtained from the mean daily prices published in the «Boletín de la Central de Anotaciones en Cuenta» edited by the Banco de España²³.

A summary of the results is shown in Table 4. Although there were some significant differences between actual and estimated returns corresponding to a particular quarter²⁴ the usefulness of the model, particularly in the field of portfolio performance, comes out when we compute the mean actual return of the whole set of portfolios and compare it to the mean return estimated by [26]. The former was 12.07% whereas the latter was 11.71%, i.e., a difference of 36 basis points²⁵.

TABLE 4

Comparison between actual and estimated mean bond portfolio returns

Actual quarter returns were obtained from randomly chosen portfolios including at least six different bonds traded in the Spanish Public Debt market. Estimated quarterly returns were computed using [26]. The total number of data was 35 in the full sample and 33 in the reduced sample. The latter excludes the data corresponding to the quarter from July 1993 to September 1993 when the EMS crisis took place.

	Full sample (July 1993- June 1996)		Reduced sample (October 1993 - June 1996)	
	Real	Estimated	Real	Estimated
Mean ¹	12.07	11.71	11.16	11.15
Standard deviation	11.13	10.42	10.81	10.47
Correlation coefficient	0.97		0.98	

¹ Mean quarterly returns on a yearly basis.

²³ As before, when an outstanding bond doesn't trade at the beginning of the quarter we use for return calculations its theoretical price according to the corresponding TSIR estimation.

²⁴ These differences may be caused, among other factors, by changes in the estimated values of the dummy variable coefficient, γ (see [14]), used in the TSIR estimation which we assumed to remain unchanged, and the fact of dealing with portfolios with a low degree of diversification.

²⁵ These returns are the mean quarterly returns calculated on a yearly basis.

Moreover, if we eliminate the data corresponding to the first quarter, which can be considered an anomalous one (the EMS crisis took place during this period and it is the quarter with the lowest number of data available) the results are extremely satisfactory. During the period from October 1993 to June 1996 the actual mean quarter return (on a yearly basis) was 11.16% and the mean quarter return according to our model was 11.15%, only one basis point below, an obviously encouraging result.

This outcome is very important for measuring the quality of portfolio management, since it can give us a good hint of what should have been the mean return of a fixed income portfolio with a given level of risk (measured by a duration vector) for a long enough time interval. Mean returns greater than those suggested by the model would suggest a good portfolio management and vice versa.

6. Conclusions

During the period covered by this study, the term structure of interest rates has experienced a sharp twist which makes traditional effective duration inadequate for measuring interest rate risk. So, we propose a two-factor model that allows to quantify the impact of a wide range of term structure movements on the price and returns of fixed income portfolios.

Usually spot interest rates are used as state variables to describe the behavior of the whole TSIR, but most previous studies have chosen these spot rates in an arbitrarily manner. In this paper, we have applied the methodology suggested by Elton, Gruber and Michaelly (1990) to select the best spot interest rate to be used for describing the Spanish TSIR according to the composition of the major Spanish mutual funds specialized in Public Debt assets.

Our results show that the three-year spot rate is the best proxy of the unknown factor for describing the TSIR in one factor models while two-month and three-year spot rates do best in a two factor model context. As in EGM, two-factor models outperformed all one-factor models (including Macaulay' duration model). An interesting result (differing in this point from EGM) is that our closed version of the two-factor model performed nearly as well as the optimal one. This may be due to the different way we modeled the functions chosen to describe spot interest rates sensitivities to the factor proxies.

A duration vector measure was developed in order to quantify the impact of interest rate shifts on portfolio value. The model was used to measure the effects of a wide range of TSIR movements on the returns of fixed income portfolios. The accuracy of the model is tested with data from the Spanish Public Debt market. Although further evidence is needed, the model seems to work very well, so it could be a useful instrument to measure portfolio performance as well as an interesting tool for asset and liability management.

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Resumen

En este artículo se desarrolla un modelo bifactorial de la estructura temporal de los tipos de interés compatible con un amplio abanico de movimientos de la estructura temporal como los observados en 1994 en el mercado español de deuda pública. Primero, se determinan los tipos de interés al contado a utilizar como proxies de los factores desconocidos, comparando el modelo resultante con otros alternativos. Posteriormente, se desarrollan dos aplicaciones del modelo para medir el riesgo de precio (mediante un vector de duraciones) y el impacto de variaciones de los tipos de interés en los rendimientos de carteras de renta fija.