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NON-LINEAR DYNAMICS AND CHAOS IN THE SPANISH STOCK MARKET*

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This paper explores the existence of non-linear dynamics and chaos in the Spanish stock market. Using the BDS test we find that the returns series shows significant linear and non-linear dependence. We also find that the dimension estimates are similar to those of processes affected by some level of noise, so that pure low dimensional chaos cannot be inferred. We show that dependence in mean is weak and that conditioning only on this kind of information cannot improve nonparametric forecasts over the random walk. Finally, we demonstrate that GARCH models cannot account for all the nonlinearity found.

1. Introduction

Testing the hypothesis of the stock exchange markets' efficiency is a recurrent issue with a long tradition (Fama, 1970). Under the hypothesis of weak efficiency, the price of an asset (P_t) completely reflects all the public information available, and its evolution is usually described as a random walk $log\ P_t = log\ P_{t-1} + \mu + u_t$, where μ is a constant trend (maybe equal to zero) and u_t are random variables independent and identically distributed (iid). If we define the rate of return in t as $r_t = log\ (P_t/P_{t-1})$, then we have $r_t = log\ P_t - log\ P_{t-1} = \mu + u_t$. In words, the hypothesis that the log prices follow a random walk implies that the rates of returns are iid.

To verify this hypothesis an easy (and common) starting point would be to calculate the autocorrelation function and verify that the coefficients corresponding to each of the lags are not significantly different from zero. However, as noted by Fama (1970), the exclusive use of the autocorrelation function only permits to indentify *linear* dependencies between the variables, thus, a plain correlogram only sheds evidence against linear relations but it does not imply the absence of dependence.

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The remaining dependence, if it exists once removed the linear one through a linear filter, could be attributed, broadly speaking, to one of three causes: a) a purely deterministic non-linear process, b) a random non-linear in variance process (for example, an ARCH-type process) or a stochastic process displaying dependence in higher conditional moments, and finally c) a non-linear in mean process affected by random perturbations. The verification of the existence of remaining dependence and, in that case, the identification of the true cause, (and of the relative importance of random and deterministic factors) is a relevant question with practical consequences for forecasting purposes.

A simple example of a situation corresponding to (a) is provided by the logistic function: $x_t = \mu x_{t-1} (1 - x_{t-1})$, $\mu \in \mathbb{R}$. For $\mu = 4$, neither its correlogram nor any other traditional econometric technique (the frequency histogram, or the Fourier spectrum, for example) can distinguish its behavior from that of white noise. Morever, the slightest observational error is amplified with time, giving a completely different dynamical evolution and suggesting, again, an erratic behavior. Informally, when we simultaneously have a purely deterministic data generating process (DGP) with an apparent random behavior and that it is sensitive to initial conditions, it is common to speak of *deterministic chaos* or simply of *chaos*.

The example provided by the logisitic equation teaches us that the randomness of stock returns could be only apparent and that some sort of neglected non-linear dynamic structure could be exploited. If this were the case, the efforts should be directed towards identifying the specific form of the underlying DGP of prices (a quite difficult task) and developing the theory for this counterintuitive behavior of markets.

Some plausible models capable of exhibiting chaos have been proposed (see Benhabib, 1992, for example). Nevertheless, the peculiarities of economic time series (high noise to amplitude ratio, small sample sizes, etc.) make the verification of the existence of chaos a difficult matter subject to many criticisms (e.g. Liu *et al.*, 1991; Ramsey *et al.*, 1990). These difficulties should not lead to abandon this effort but suggest that one should be especially careful in the interpretation of the results.

Some relevant work discussing the evidence of non-linear structure in different time series are Frank and Stengos (1989), Blank (1990) and DeCoster *et al.* (1992) for futures commodities prices; Hinich and Paterson (1985), Scheinkman and LeBaron (1989) and Hsieh (1991) for stock returns; Hsieh (1989) on exchange rates; Barnett and Chen (1988) and DeCoster and Mitchell (1991) on monetary agregates and components; Frank and Stengos (1988a), Frank *et al.* (1988), Brock and Sayers (1988) and Scheinkman and LeBaron (1989) on macroeconomic data, and Bajo *et al.* (1992) in the Spanish case. To summarize, we can say that the evidence of deterministic chaos found along these papers is mixed.

In this debate to determine whether the «world» is of an intrinsically linear nature affected by random shocks or purely deterministic and non-linear there is, still, some room for a less belligerent position: considering that economic time series are generated by DGP's with linear and/or non-linear structure affected by random perturbations. Obviously, depending on the relative level of the random perturbations with respect to the deterministic component and the precision required, a certain time series could be catalogued as purely random (with an infinite number of degrees of freedom) or stochastic («reasonably» described by a finite, and posibly small, number of degrees of freedom). In any case, in our view, at the present state of things one should not expect definite results, since the theory of chaos relevant for economic time series analysis is scarce and only permits essentially qualitative conclusions.

In this paper we precisely adopt this position, concluding that while the hypothesis of the random walk model can be strongly rejected (for the stock returns series studied), one of its alternatives, the purely chaotic explanation, cannot be maintained, and suggesting that the magnitude of the random component makes the deterministic component unimportant for forecasting purposes.

The structure of the paper is a follows: in Section 2 we briefly introduce some concepts of chaos theory and the BDS statistic for testing non-linear dependence. In Section 3 we describe the database used and the results obtained, discussing the evidence in favor and against the hypotheses of nonstationarity and nonlinearity in mean and variance. Section 4 summarizes some conclusions and proposals for future work.

2. Theoretical setup

In this section we will briefly review some basic concepts related to Chaos Theory and we will also introduce other concepts needed for our subsequent exposition. For additional details refer to Brock *et al.* (1991) and Grassberger and Procaccia (1983).

2.1. *Some basic concepts on chaotic processes*

Instead of attempting a formal definition of chaos, which remains controversial, we will try to illustrate the basic features associated with it. See Lorentz (1989) and Brock *et al.* (1991) for a more formal treatment.

One of the most important features of chaos is *ergodicity*, which means that long term trajectories starting at a typical initial value «fill» the complete range of values (in the sense that they enter any interval in that range), and that the visiting frequency of any interval is independent of this particular initial value. Figure 1 illustrates this feature for the above mentioned logistic function, by showing two frequency histograms for the first 50.000 observations starting at initial values 0.1 (a) and 0.7 (b), where the unit interval has been subdivided in 100 equal length subintervals. Note that any subinterval is visited with approximately the same frequency for any of the two initial conditions.

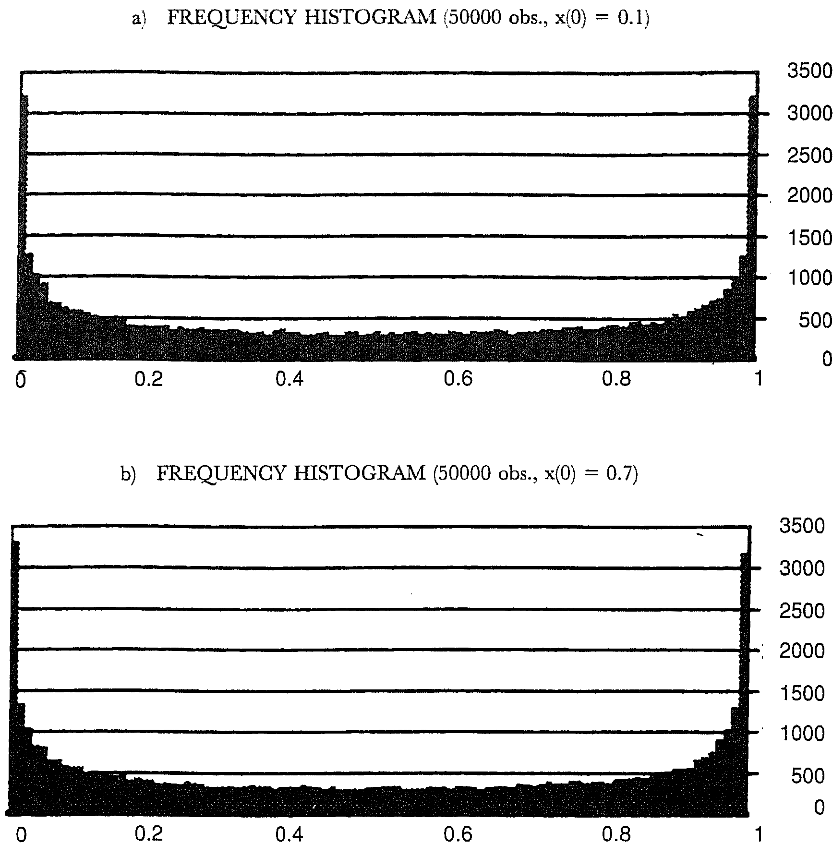


Figure 1
Ergodicity

Another fundamental characteristic of chaotic processes is their *sensitive dependence to initial conditions,* which means that trajectories starting at two arbitrarily close initial points will eventually diverge. Figure 2 illustrates this behavior for the logistic function. In the upper part we have plot two series of the first 100 iterates of the logistic function with the initial conditions $x_0 =$ 0.1 and $x_0 =$ 0.100001 ($\{x_t\}$ and $\{y_t\}$, respectively) while in the lower part we have plotted the difference $\{x_t - y_t\}$ between them. As one can see, during the first periods the evolution of the variables is indistinguishable so that their difference is negligible, but after some periods (around 15) this difference is amplified with time, giving a completely dissimilar behavior. Also note that the divergence does not exhibit any pattern, therefore, it can not be corrected. What all this means is that even some purely deterministic processes can not be accurately predicted after a short while even in the presence of slight observational errors.

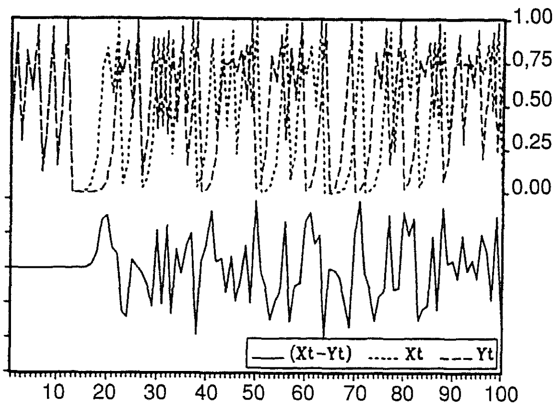


Figure 2
Sensitivity to initial conditions

As a consequence of ergodicity and sensitivity to initial conditions, chaotic time series tend to look random and indeed they are very similar to stochastic white noise. Figure 3 shows the estimated autocorrelogram (50 lags) for a series of 1000 observations (a common size in economic data) of the logistic. Note that only the autocorrelation coefficient at the lag 48 is significant. In the light of this result one would be tempted to reject any kind of dependence.

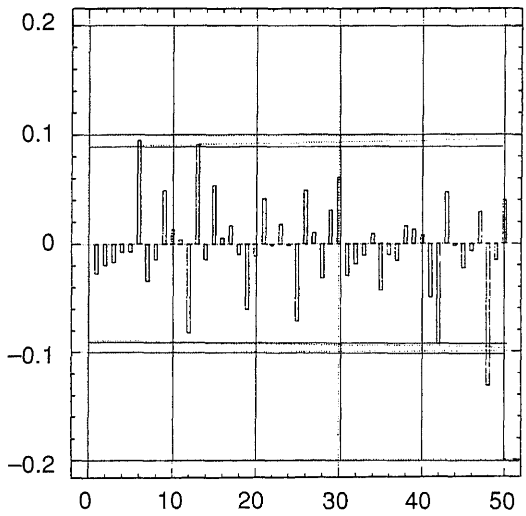


Figure 3
Autocorrelation coefficients

2.2. *Time series characteristics associated to a chaotic process.*

In Section 2.1. we have briefly illustrated, through an example of a chaotic process, the apparent randomness of chaotic time series. However, as it is the case of this paper, sometimes one is confronted with a given time series apparently random and the problem is to determine whether the underlying process is chaotic. In order to tackle this problem, we need to consider characteristics of the time series which imply evidence of chaos in the underlying process. Three of the most important tools useful for this analysis are the Lyapunov exponents, the Kolmogorov entropy and the correlation dimension.

a) Lyapunov exponents and Kolmogorov entropy

Lyapunov exponents (Oseledec, 1968), can be viewed as a generalization of the familiar eigenvalues of a matrix (for a difference equation system with a stable fixed-point, they would be the logarithm of the absolute values of the eigenvalues of the Jacobian matrix evaluated at the fixed point). They measure the divergence, in every direction, of two trajectories starting at nearby points in the phase space. A negative Lyapunov exponent means an approach of these trajectories in the corresponding direction, while a positive exponent means that the distance grows. Sensitivity to initial conditions is implied by the existence of, at least, a positive Lyapunov exponent, so that a test based on Lyapunov exponents would be a direct test for chaos. Unfortunately, two main problems arise on using Lyapunov exponents to detect a chaotic process: first, their computation is extremely inaccurate and second, there is no sampling theory to test for the significance of the estimates.

The Kolmogorov entropy (Sinai, 1976) is a lower bound on the sum of the positive Lyapunov exponents and it measures the mean rate of information creation. For a completely random process, the Kolmogorov entropy is infinite, while for a chaotic system it is bounded. In contrast with Lyapunov exponents, some asymptotic results for its distribution are available (see Baek and Brock, 1992) so that it is possible to conduct tests based on this measure. As we will not use these two concepts in this paper, we do not pursue the full presentation of them. The interested reader can find a more formal and detailed treatment, with economic motivations, in Frank and Stengos (1988b).

b) Correlation dimension

There is little doubt that the best known technique for the analysis of chaotic systems is the correlation dimension. This is due to the fact of an easy computation and the availability of a sampling theory. Since many of the results presented in this paper rely heavily on the use of the correlation dimension, we will give a more detailed treatment.

Let (x_t) be a sequence of observations that, for simplicity, we can assume unidimensional (the multidimensional case is treated in Baek and Brock, 1988). We will call *h-histories* to the h-dimensional vectors $x_t^h = (x_t, x_{t+1}, \ldots, x_{t+h-1})$. In words, a h-history of an observation x_t, is an h-dimensional vector, each of whose coordinates is the value of the series for future periods $t + i$ $(i = 0,1,\ldots\ h-1)$. In a geometric sense, the procedure of taking h-histories can be viewed as transforming n observations of an univariate series into n-h observations of a multivariate series of dimension h. For obvious reasons we will call h the *embedding dimension.*

We define the *correlation integral* as

$$C_h(\varepsilon, n) = \frac{2}{n_h(n_h - 1)} \sum_{t<s} I_\varepsilon(x_t^h, x_s^h)$$

where $n_h = n - h + 1$, $\varepsilon \in \mathbb{R}_+$ and $I_\varepsilon(x_t^h, x_s^h) = 1$ iff $\| x_t^h - x_s^h \| < \varepsilon$, with $\| \ \|$ the suprem norm.

Intuitively, the correlation integral is the proportion of pairs of h-histories within a distance ε. Therefore, it can be viewed as an estimator of the probability that two h-histories x_t^h, x_s^h are closer than ε, that is:

$$C_h(\varepsilon, n) \rightarrow \text{Prob}\ \{|x_{t+i} - x_{s+i}| < \varepsilon, \forall\ i = 0,1,2,\ldots, h-1\} \text{ as } n \rightarrow \infty$$

The *correlation dimension* for the embedding dimension h is defined as:

$$C_h = lim_{\varepsilon \to 0}\ lim_{n \to \infty} \frac{log\ C_h(\varepsilon, n)}{log\ (\varepsilon)} \qquad [1]$$

Another slightly different definition of the correlation dimension is:

$$C_h = lim_{\varepsilon \to 0}\ lim_{n \to \infty} \frac{\partial\ log\ C_h(\varepsilon, n)}{\partial\ log\ (\varepsilon)} \qquad [2]$$

which can be approximated, for ε small, $\varepsilon' < \varepsilon$, and n large by:

$$lim_{\varepsilon \to 0} \frac{log\ C_h(\varepsilon', n) - log\ C_h(\varepsilon, n)}{log\ (\varepsilon') - log\ (\varepsilon)}$$

The meaning of [2] is the limiting slope, as ε tends to zero and n tends to infinity, of $log\ C_h(\varepsilon, n)$ versus $log\ \varepsilon$ or, in other words, the limiting value of the elasticity of $C_h(\varepsilon, n)$ with respect to ε.

It can be shown (Grassberger and Procaccia, 1983) that for a deterministic (chaotic) process then $C_h = k$ if $h > k$, being k a constant, while for stochastic white noise $C_h = h$ for all h. This means that, from a practical point of view, the only thing we have to do is to calculate the correlation dimension for increasing values of h. If the process is deterministic then C_h will stabilize for certain h while if the process is purely stochastic it will continue to increase with h.

Figure 4 helps to give an intuitive view of the correlation dimension. First (part a), let us suppose that we have observations generated from the logistic model $x_{t+1} = 4\, x_t\, (1 - x_t)$ and consider the space of 2-histories of this series. Suppose now that we fix any ε and calculate the average proportion $p\,(\varepsilon)$ of 2-histories lying inside an ε-square (equivalent, for our purpose, to calculate the proportion of pairs of 2-histories closer than ε). If we now half the distance ε, it is easy to see that the considered proportion also halves. More generally stated, a reduction of ε by a r factor will reduce $p\,(\varepsilon)$ by the same r factor. It is clear that this behavior will hold, as long as the number of points tends to infinity, for sucessive iterations of this reduction procedure, leading to a correlation dimension

$$C_2 = lim_{q \to \infty}\, lim_{n \to \infty} \frac{log\,(r^q p\,(\varepsilon))}{log\,(r^q \varepsilon)} = lim_{q \to \infty} \frac{log\,(r^q\, p\,(\varepsilon))}{log\,(r^q \varepsilon)} =$$

$$= lim_{q \to \infty} \frac{qlogr + logp\,(\varepsilon)}{qlogr + log\,\varepsilon} = 1$$

which is the intuitively expected dimension of a curve.

Consider now a purely random process $x_t \sim U_{[0,1]}$ in which observations are uniformly distributed along the interval [0,1]. In this case (part b), the 2-histories «fill» a square and a reduction of ε by a r factor will reduce $p\,(\varepsilon)$ by the factor r^2. Thus, as long as the number of observations tends to infinity, sucessive iterations of this reduction procedure will lead to a correlation dimension:

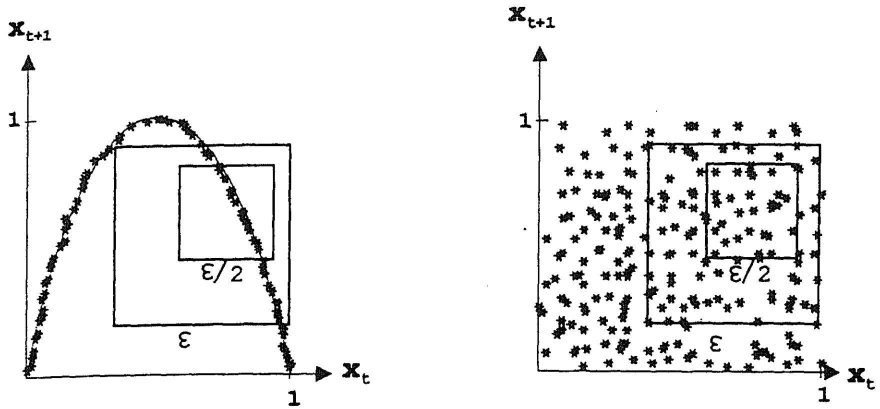
$$C_2 = lim_{q \to \infty}\, lim_{n \to \infty} \frac{log\,(r^{2q} p\,(\varepsilon))}{log\,(r^q \varepsilon)} = lim_{q \to \infty} \frac{log\,(r^{2q}\, p\,(\varepsilon))}{log\,(r^q \varepsilon)} =$$

$$= lim_{q \to \infty} \frac{2qlogr + logp\,(\varepsilon)}{qlogr + log\,\varepsilon} = 2$$

which is the expected dimension of a square.

For a random process like strict white noise, it is easy to see that, for every embedding dimension h, the sequence of h-histories tend to fill an h-cube, leading by the above reasoning to a correlation dimension $C_h = h$. However, the correlation dimension of the logistic process is 1 for every embedding dimension. It is worth noting that, in general, the correlation dimension of a process is not an integer (see Brock *et al.* (1991)).

While the computation of the correlation dimension is certainly easy, two problems arise. The first of them is the choice of the parameter ε. If ε is too large, then all the points are closer than ε, giving a constant value of $C_h\,(\varepsilon,n)$ for every h. On the other hand, if ε is too small, then the correlation integral will capture too few points giving inaccurate estimates.



a) Logistic Equation sequence of 2-hist. b) Random sequence of 2-hist.

Figure 4
Correlation dimension

The second problem is related to the sample size. In fact, as we increase h, we will have fewer and fewer non-overlapping h-histories, so that, for samples of moderate size, we will be able to characterize only low dimensional chaos, that is, DGP of small dimension (say six or less). In this paper we will give values of C_h (ε,n) for different values of ε and h to verify the robustness of the estimates.

2.3. *Testing for nonlinearity: the BDS test*

As we showed in Section 2.1, purely deterministic non-linear models can exhibit a complex behavior that appears as random. Of course, there are also other non-linear stochastic alternatives capable of displaying that behavior such as the nonlinear moving average model (Robinson, 1979), the threshold autoregressive model (Tong and Lim, 1980), the bilinear model (Granger and Andersen, 1978) and (notably) the autoregressive conditional heteroskedasticity (ARCH) model (Engle, 1982), to mention a few.

The discovery that purely deterministic and non-linear stochastic models behave as random models notably complicates the process of hypothesis testing and model building. This is probably due to the fact that there is not a general test capable of detecting any of the above mentioned processes. In general one employs a test which has known power only against a specific alternative (for example, a test against threshold nonlinearity as in Tsay, 1989), so that failing to reject the null does not allow to discard any other possible alternative.

Some general tests for nonlinearity have been proposed in the literature. For example, Keenan (1985) introduced a test based on the correlation of residual errors of a linear regresion and the linear forecast, making it posible to detect if the squared forecast has aditional forecasting ability as well as departures from linearity in mean. Tsay (1986) proposes an extension of Keenan's test by evaluating the statistical significance of the crossterms of a series after purging any linear dependence. This test has good power against the non-linear moving average and the bilinear models but its performance is poor against ARCH alternatives. McLeod and Li (1983) show that the autocorrelation coefficients and the Box-Pierce statistic of the squared residuals from an ARMA model are useful to test non-linear dependence. Another alternative is to use the bispectrum as in Subba Rao and Gabr (1980), a major weakness of this test is that it has low power against processes with zero third-order cumulants such as ARCH. The Engle's test for ARCH (Engle, 1982) is not only near optimal to detect heteroskedasticity but has also good power to detect many types of nonlinearity. Finally, Robinson (1991) proposes a new class of tests based on the Kullback-Leibler information criterion to check for independence.

The above list is by no means exahustive, it only sketches some of the best known approaches to test for nonlinearity. Recently, Brock, Dechert and Scheinkman (1987) have suggested a new test to verify the hypothesis that the observations of a time series are iid. The Brock-Dechert-Scheinkman (BDS in what follows) test allows to detect departures from the iid hypothesis caused by non stationarity, nonlinearity or the existence of chaos by establishing a measure of dependence based on the correlation integral.

BDS have noted that under the hypothesis that x_t are independent then we have:

$$C_h(\varepsilon,n) \rightarrow \pi_{i=0}^{h-1} Prob\{| x_{t+i} - x_{s+i} | < \varepsilon\} \quad \text{as} \quad n \rightarrow \infty,$$

and if x_t are identically distributed then also:

$$C_h(\varepsilon,n) \rightarrow C_1(\varepsilon,n)^h, \text{ with probability one, as } \quad n \rightarrow \infty.$$

BDS have also shown that, under the iid null, $n^{1/2}\{C_h(\varepsilon,n) - C_1(\varepsilon,n)^h\}$ has an asymptotic distribution $\mathcal{N}(0,\sigma_h(\varepsilon))$, where $\sigma_h(\varepsilon)$ can be consistently estimated, so that it is posible to construct tests against the iid hypothesis for any significance level. Note that although our primary concern is to test for (non-linear) dependence, the rejection by the BDS test is also consistent with nonstationarity.

As this test is based on the correlation integral similar problems arise in choosing ε and h. Brock *et al.* (1991) recommend a value for ε in the range of one-half to two times the standard deviation of the data. The choice of h depends on the sample size and, as we will show, $h = 8$ seems the highest value one can use in the most optimistic case. Brock *et al.* (1991) also show that the test has good size and power against a variety of linear and non-linear alternatives, even in samples of moderate size. In addition, the BDS test is able to detect more general types of nonlinearity than the Engle or Tsay tests; for example an ARCH-type process in which the conditional variance depends on its past values in a non-linear fashion.

3. Empirical results

The database used is composed of 1.256 observations (x_t) (from 2 January 1989 to 31 January 1994) of the General Index of Madrid's Stock Market, not corrected for dividends or splits. The augmented Dickey-Fuller test failed to reject the null of a unit root in the log prices series (a common result in many economic series as Nelson and Plosser (1982) noted). As the Dickey-Fuller tests have low power against specific alternatives (see, for example DeJong *et al.* (1989) and Diebold and Rudebusch, (1990)) it is also interesting to take stationarity as the null.

Kwiatkowski *et al.* (1992) have suggested to take the residuals from the regression of x_t on an intercept and a time trend and compute the statistic $\eta(q) = n^{-2} \Sigma S_t^2 / s^2(q)$, where S_t is the partial sum of the residuals, $s^2(q)$ is a consistent estimator of the long-run variance for truncation lag q (see Philips and Perron, 1988) and n is the sample size. They also demonstrate that under the nulls of trend or level stationarity η converges weakly to a Brownian bridge, so that it is possible to construct tests for any significance level.

Following Kwiatkowski *et al.* (1992, p. 174) we have computed the η statistic considering the regression of x_t on an intercept (level-stationarity) and both an intercept and a trend (trend-stationarity) with $q \leq \text{int}\ [8(n/100)^{1/4}]$. We found values of $\eta(q) \geq 1.88$ and $\eta(q) \geq .727$ (5% critical values are .463 and .146, respectively) implying the rejection of both types of stationarity. These results combined with the ones obtained by the augmented Dickey-Fuller test permit to be confident with the existence of a unit root. At the light of the ADF, we achieved stationarity after taking first differences.

Table 1 reports some summary statistics of the price levels and returns series and shows some caracteristics common to many financial series: the unconditional distribution is clearly leptokurtic and the skewness coefficient also differs from the one corresponding to a normal distribution, in adition the LM test for joint normal kurtosis and skewness also rejects the normality hypothesis.

Following Brock *et al.* (1991) we present the BDS statistic for values of ε between one-half and two times the standard deviation and embedding dimensions between 2 to 8 (Table 2). The hypothesis of iid variables is rejected in all the cases with a confidence level of 1%. Our results are quite similar to those of Hsieh (1989) who employs exchange rates and to those of Scheinkman and LeBaron (1989) and Hsieh (1991) for stocks prices.

For illustrative purposes, we have also created two artificial series of the same size. The first one is composed of iterations of the logistic function with

TABLE 1
Summary statistics

	Levels	Returns
Sample size (n)	1256	1255
Mean	266.36	0.000207
Median	268.81	0.000085
Standard dev	33.07	0.0102
Minimum	179.48	−0.0861
Maximum	358.31	0.0633
Skewness	−0.136	−0.52920
		(0.06914)
Kurtosis	−0.57	8.88402
		(0.13828)
LM test		1868.99*

Note: Standard errors in parentheses are computed as follows: $(6/n)^{1/2}$ for the coefficient of skewness (S), $(24/n)^{1/2}$ for the coefficient of kurtosis (K).
The Lagrange Multiplier test (LM test) is defined as n [$S^2/6 + (K-3)^2/24 \sim \chi^2_2$, the critical value at the 5% level is 5.99.
(*) Significant at the 5% level.

TABLE 2
BDS Statistic
(1.255 observations)

	1/2 σ			σ		
h	logistic	uniform	returns	logistic	uniform	returns
2	544.18	−3.19	10.37	222.22	−0.295	9.390
3	712.19	2.90	13.21	214.43	−0.321	11.17
4	924.10	2.83	16.43	205.30	−0.359	13.13
5	1271.1	3.16	19.83	202.51	−0.295	14.58
6	1845.6	5.74	25.75	208.36	−0.170	16.97
7	2827.6	1.07	34.05	217.39	−0.363	19.46
8	4662.7	−11.89	45.68	235.54	0.262	22.33
	3/2 σ			2σ		
h	logistic	uniform	returns	logistic	uniform	returns
2	14.40	0.215	9.020	−24.27	0.146	8.640
3	3.340	−0.015	10.56	−19.13	−0.121	10.40
4	2.930	−0.278	12.01	−15.57	−0.439	11.51
5	1.240	−0.667	12.79	−12.97	−0.745	12.05
6	1.990	−0.671	13.92	−11.64	−0.638	12.70
7	24.78	−0.737	14.89	−10.33	−0.599	13.12
8	−26.23	−0.813	15.91	−0.390	−0.567	13.56

Note: BDS statistic for values of ε equal $1/2\sigma$, σ, $3/2\sigma$ and 2σ, where σ is the standard deviation of the series. σ(logistic) = 0.3519; σ(uniform) = 0.284; σ(returns) = 0.0102. Critical values: 1.645 (10%); 1.960 (5%); 2.326 (2%); 2.576 (1%). Underlined values denote an erroneous rejection of the nul hypothesis.

$\mu = 4$ and $x_0 = 0.1$, where the first 200 transients have been eliminated. The second is created from randon sampling with replacement from a uniform distribution in [0,1]. As we can see, the BDS rejects the iid hypothesis for the logistic in practically all the cases. The test is worse for the uniform, for $\varepsilon = 1/2\sigma$ it wrongly rejects the null at the 1% for six of the seven values of h. In the rest of the cases the test behaves as expected. It should be noted, however, that for the noisy case, the range $[1/2\sigma, 2\sigma]$ seems over-optimistic. For example, for $\varepsilon = 1/2\sigma$ and $h = 8$ there are only fifteen pairs of points closer than ε so the value for the BDS test (−11.89) is not reliable.

As we previously mentioned, the BDS has power to detect the presence of structure (linear or not) in the data, so that rejecting the iid hypothesis does not automatically imply accepting the one of non-linear dependence. To see if the existence of linear structure is the cause of the rejection, we have calculated the autocorrelation coefficients for the first 50 lags (Table 3). The presence of autoregressive conditional heteroskedasticity can affect the identification process, so the coefficients and the Box-Pierce statistic reported have been corrected as in Diebold (1988). As it can be seen, the coefficient for the first lag is significantly different from zero, probably due to a Fisher's effect.

TABLE 3
Autocorrelation coefficients and Box-Pierce statistic
(heteroskedasticity consistent)

j	ρ_j	j	ρ_j
1	0.142060* (0.047769)	12	0.060041 (0.038851)
2	0.040925 (0.047121)	13	0.040189 (0.035787)
3	0.031077 (0.033659)	14	−0.00969 (0.041974)
4	0.001845 (0.032834)	15	0.021735 (0.038108)
5	0.021772 (0.040730)	20	−0.02685 (0.032766)
6	0.035018 (0.034498)	25	0.041729 (0.036236)
7	0.041111 (0.033822)	30	−0.00919 (0.029354)
8	0.016290 (0.034533)	35	0.049450 (0.029511)
9	−0.01359 (0.034824)	40	−0.02764 (0.029065)
10	0.035219 (0.039332)	45	0.001286 (0.029948)
11	−0.00906 (0.034031)	50	−0.00865 (0.030006)

Note: Standard errors in parentheses.
(*) Significant at the 1% level. Box-Pierce statistic 70.50. Box-Pierce (corrected): 42.38

TABLE 4
Autocorrelation coefficients and Box-Pierce statistic
for the AR(1) residuals and squared AR(1) residuals of the returns series

	ρ_j			ρ_j	
j	residuals	squared resid.	j	residuals	squared resid.
1	−0.00338	0.258919*	12	0.058272	0.076918*
2	0.017596	0.157226*	13	0.034560	0.064748*
3	0.026102	0.041238	14	−0.019300	0.092924*
4	−0.00551	0.038289	15	0.021593	0.064088*
5	0.017534	0.096294*	20	−0.02740	0.023087
6	0.027187	0.049772	25	0.045616	0.054833
7	0.035335	0.037454	30	−0.00974	−0.00916
8	0.012833	0.051330	35	0.049668	−0.00912
9	−0.02141	0.056198	40	−0.02300	0.004928
10	0.040098	0.076270*	45	−0.00028	−0.01459
11	−0.02336	0.053650	50	−0.01423	0.002808

Note: Box-Pierce statistic: AR(1) residuals: 42.73; squared AR(1) residuals: 201,94*.
(*) Significant at the 5% level.

To remove linear dependencies, we fitted ARMA (p,q) models with p,q = 0,1,2,...,5 to the series, selecting the lag structure (p,q) which minimizes the Akaike Information Criterion. The model chosen was an AR(1), which we employed to filter the returns series. In Table 4 (columns 1 and 3) the efficiency of the filtering is evident: the correlation coefficients are not significantly different from zero, and the Box-Pierce statistic allows to reject the hypothesis of joint autocorrelation in the residuals. In the second and fourth columns we have calculated the same statistics for the squared residuals. Many of them are significantly different from zero, suggesting the existence of remaining non-linear structure.

TABLE 5
BDS Statistic for the AR(1) residuals of the returns series

h	$1/2\sigma$	σ	$3/2\sigma$	2σ
2	9.570	8.800	8.850	9.400
3	12.90	10.79	10.35	11.00
4	16.51	12.98	11.91	12.16
5	20.14	14.60	12.76	12.67
6	26.34	17.14	13.97	13.29
7	35.72	19.87	14.96	13.66
8	49.68	23.06	16.01	14.05

Note: BDS statistic for values of ε equal $1/2\sigma$, σ, $3/2\sigma$ and 2σ, where σ is the standard deviation of the residuals ($\sigma = 0.0101$). Critical values: 1.645 (10%); 1.960 (5%); 2.326 (2%); 2.576 (1%).

Brock (1987) has demonstrated that the asymptotic distribution of the BDS statistic in not affected if the residuals of an adjusted linear model are used instead of the raw data (this is also true for many non-linear models, with the notable exception of ARCH-type models). Based on this result, we have calculated the BDS statistic for the AR(1) residuals (Table 5). As it can be seen, we again reject the hypothesis of iid in all the cases with a confidence level of 1%. As the linear components have been previously removed, the only two alternatives of the null are the nonstationarity or nonlinearity (stochastic or deterministic) of the residuals.

3.1. Nonstationarity

The nonstationarity hypothesis implies a changing DGP. To verify that structural changes are not responsible for the rejection, we will employ two different procedures. First we consider that if the process was stationary each of the halves of the series should have the same empirical density as the whole series. To check for this we divide the range of the whole series into k intervals and evaluate the number of points falling into these intervals (n_k):

$$P_k = \frac{n_k}{\sum_k n_k}$$

A natural way to compare this empirical density with the one obtained for each of the halves of the series is to construct a χ^2 test:

$$D = \sum_k \frac{(n_k^{1/2} - Zp_k)^2}{Zp_k} \sim \chi^2_{k-1}$$

where $n_k^{1/2}$ is calculated for each of the halves, and $Z = n/2$. The choice of the number of intervals is somewhat arbitrary, Hsieh (1988) used values in the range [20, 50] but in this work we will follow a suggestion of Isliker and Kurths (1993) that consists on using an initial value of $k = 90$ and merging intervals whenever $Zp_k < 5$. In Table 6 we show the results. As one can see, the null hypothesis of equality of distributions can not be rejected at the 5% level for any of the halves of the series.

TABLE 6
Chi-squared test for stationary

Sample	Subsample	D	χ^2_{k-1}	k_{init}	$k = k_{end}$
1-1255	1-627	24.0	32.7	90	22
	628-1255	24.5	32.7	90	22

Note: Chi-squared test for identical empirical densities. Values of χ^2 corresponding to a 5% level test

Our next approach consists on a heuristic procedure suggested by Hsieh (1991). Hsieh argues that the problem of nonstationary is less important for high frequency data sampled during shorter periods since structural changes become less plausible. Following this we have calculated the BDS statistic for data sampled at different frequencies. First, and for reasons of completeness, we have built a weekly returns series, taking the first trading day after a Sunday as the reference. This series passed the augmented Dickey-Fuller test for stationarity and a significant autoregressive linear structure was not present.

TABLE 7
BDS Statistic
Weekly Returns
(265 observations)

h	$1/2\sigma$	σ	$3/2\sigma$	2σ
2	2.13*	3.07 **	4.03**	4.84**
3	2.54*	3.56 **	4.35**	4.73**
4	3.62**	4.42 **	4.70**	4.61**
5	3.31**	4.66 **	4.70**	4.38**

Note: BDS statistic computed for values of ε equal to $1/2\sigma$, σ, $3/2\sigma$, 2σ, where σ is the standard deviation of the series ($\sigma = 0.0268$). Critical values: 1.645 (10%); 1.960 (5%); 2.576 (1%).
(*) significant at the level.
(**) significant at the 1% level.

When applied to the weekly returns series, the BDS statistic also rejects the null in all the cases with a confidence level of 5%, and in all but in two cases at the 1% level (Table 7). It shoud be noted, however, that the values of the statistics are significantly smaller than the ones obtained for the daily returns, suggesting differences in the DGP for different frequencies. It is also quite important to note that the small sample size does not allow to be completely confident with the estimates in all the cases. For $h = 5$ there are only $265/5 = 53$ non-overlapping h-histories which are used to determine if they completely fill a 5-dimensional space.

TABLE 8
BDS Statistic for the 15 minutes returns of the IBEX 35
four quarterts of the year 1993

1st. quarter					2nd. quarter				
h	$1/2\sigma$	σ	$3/2\sigma$	2σ	h	$1/2\sigma$	σ	$3/2\sigma$	2σ
2	11.90	11.08	9.90	7.36	2	8.20	9.59	9.59	9.25
3	15.38	13.06	10.75	7.81	3	11.14	11.69	11.14	10.43
4	18.60	14.60	11.25	8.14	4	13.95	12.99	11.47	10.49
5	22.42	15.65	11.42	8.04	5	16.00	13.69	11.23	10.04
6	26.36	16.41	11.50	7.94	6	18.22	14.20	11.05	9.59
7	31.65	16.92	11.32	7.67	7	21.04	14.55	10.74	9.16
8	38.30	17.37	11.11	7.38	8	25.00	15.20	10.59	8.80

TABLE 8 (2)
BDS Statistic for the 15 minutes returns of the IBEX 35
four quarterts of the year 1993

3rd. quarter					4th. quarter				
h	$1/2\sigma$	σ	$3/2\sigma$	2σ	h	$1/2\sigma$	σ	$3/2\sigma$	2σ
2	8.21	8.71	7.78	7.18	2	5.93	6.43	5.12	4.95
3	10.80	10.74	9.18	7.99	3	7.79	7.93	6.20	5.38
4	13.67	12.16	9.97	8.63	4	8.77	8.25	6.07	5.01
5	16.07	13.00	10.26	8.74	5	9.83	8.64	6.11	4.75
6	19.21	13.73	10.28	8.60	6	10.74	9.01	6.25	4.53
7	22.74	14.54	10.31	8.38	7	11.70	9.26	6.31	4.40
8	26.68	15.37	10.34	8.13	8	12.26	9.23	6.24	4.32

Note: BDS for residuals from a AR(2), AR(1), AR(10) and AR(6) of the 15 minutes returns series for the 1st., 2nd., 3rd. and 4th. quarters of 1993, respectively. Number of observations: quarters 1st. (1549), 2nd. (1549), 3rd. (1649). 4th. (1324). Critical values: 1.645 (10%); 1.960 (5%); 2.326 (2%); 2.576 (1%).

To see if the behavior of the BDS test is the same for shorter frequencies it would be desirable to have observations of the same index sampled more frequently. Unfortunately, some of the values included in the General Index are not continously traded so we have to use another index. We will make use of the IBEX35 index which shows a high correlation (around a 99,26% for returns for the year of reference) with the General Index, previously employed. Our database will consist of 6.075 observations of the IBEX35 sampled every fifteen minutes during the year 1993. We will divide this data set into four subsets corresponding to each of the quarters. We have identified the respective linear models and computed the BDS test for the residuals. As we can see from Table 8, the hypothesis of iid returns is rejected at the 1% level for each of the series. As the time horizon is very short, a changing DGP is not plausible so we conjecture that nonstationarity is not the cause of rejection.

3.2. Nonlinearity

From the preceeding analysis, the (provisional) most plausible hypothesis is that of the presence of significant non-linear structure in the data. In this section, and using three different perspectives, we will try to investigate if a chaotic (pure deterministic underlying process) explanation is sustainable and if not if we can discriminate between the different types of stochastic non-linear dependence.

a) Scheinkman «shuffle» test

Our first analysis will be based on the correlation dimension, previously introduced. The correlation dimension is easily calculated and it has more robust properties than other estimates (Lyapunov exponents, for example) proposed in chaos theory. In any case, as we mentioned, one can use many other techniques (Kolmogorov dimension, recurrence plots, etc.) to properly characterize chaos, but we leave this for future work.

In Table 9 we have calculated the correlation dimension for $\varepsilon_j = .9^j$ with j chosen so that ε_j is, approximately, in the range between one-half and two times the standard deviation of the rescaled series. As one can see from a) (logistic) and b) (uniform), the quality of the estimates of C_h (ε, n) degrades with increasing values of h, making them unreliable for $h > 10$. In qualitative terms one can see that in the deterministic case the calculated dimension does not increase substantially for increasing values of h, while in the stochastic case this is appreciable. For the AR(1) residuals the behavior is not definitive, the calculated dimension increases at a lower rate than the expected, stabilizing at a value of around 6 or 7. As this kind of «ocular econometrics» (Liu *et al.*, 1991) does not give very reliable estimates, we will turn to a different strategy.

Scheinkman (1989) has noted that any dependence present in the data can be easily destroyed just by sampling with replacement from the original series and by building another «shuffled» series which possesses the same statistical characteristics as the original one. If one then computes the correlation dimension for this «shuffled» series this should not be, theoretically, smaller than the corresponding to the original one.

We have constructed 20 «shuffled» series and computed their correlation dimension. In Table 10a we give the maximum and minimun values obtained for each ε and h . We have underlined the values for which the original series has a value of its correlation dimension greater than the minimum of the «shuffled» ones. As one can see, almost in every case the estimates of the original series are smaller. From table 10b one can also see that in every case the dimension is never greater than the mean dimension of the «shuffled» series. For example, for $h = 8$ the dimension for the original series is around 6 and the mean for the «shuffled» series is 7. This difference is not significant because of the lack of robustness of the estimates and also because, as one can check from Table 8b, there is a constant downward bias in their estimation in the noisy case. The differences, if any, should appear in higher embedding dimensions, but due to insufficient sample size we were not able to verify this. Our conclusion is that the original and «shuffled» series show a similar level of randomness and, as a consequence, that low dimensional chaos cannot be inferred. Finally, we have to note that from a qualitative point of view the behavior of the «shuffled» series is more similar to that of a stochastic series, since the estimates collapse for smaller values of ε.

TABLE 9

Correlation Dimension for 1.255 observations of the logistic equation, uniform distribution and AR(1) residuals of the returns series

a) Logistic equation

j	ε_j	h=2	h=4	h=6	h=8	h=10	h=12	h=14	h=16
4	0.656100	1.242	2.995	5.573	8.119	10.729	13.426	15.768	17.684
5	0.590490	1.184	2.619	3.746	4.844	5.836	6.601	6.906	6.329
6	0.531441	1.044	1.693	2.212	2.670	2.968	3.039	2.905	2.775
7	0.478297	0.967	1.402	1.497	1.489	1.508	1.483	1.364	1.136
8	0.430467	0.955	1.373	1.569	1.750	1.888	2.049	1.905	1.496
9	0.387420	0.948	1.376	1.995	2.492	2.935	3.069	2.741	1.912
10	0.348678	0.951	1.416	2.095	2.778	3.365	3.197	2.498	1.206
11	0.313811	0.931	1.337	1.430	1.616	1.191	0.747	0.475	0.179
12	0.282430	0.955	1.261	1.586	1.883	1.977	1.698	0.789	0.159
13	0.254187	0.955	1.278	1.498	1.650	1.449	1.079	0.462	0.268
14	0.228768	0.879	1.133	1.289	1.328	1.147	0.589	0.239	0.023
15	0.205891	0.928	1.060	1.241	1.136	1.040	0.744	0.095	0.023
16	0.185302	0.938	1.061	1.295	1.518	1.639	0.853	0.358	0.240
17	0.166772	0.854	0.985	1.274	1.343	0.900	0.524	0.349	0.147
18	0.150095	0.937	0.985	1.09	0.964	0.846	0.433	0.127	0.024

Note: Standard dev. = σ_r = 0.3519; $1/2\sigma_r$ = 0.17595; $2\sigma_r$ = 0.7038

b) Uniform distribution [0,1]

j	ε_j	h=2	h=4	h=6	h=8	h=10	h=12	h=14	h=16
4	0.656100	0.916	1.839	2.765	3.688	4.601	5.514	6.437	7.380
5	0.590490	1.061	2.130	3.201	4.270	5.321	6.394	7.503	8.638
6	0.531441	1.244	2.500	3.788	5.065	6.315	7.563	8.811	10.127
7	0.478297	1.313	2.627	3.953	5.302	6.674	8.126	9.563	11.103
8	0.430467	1.370	2.742	4.131	5.545	6.956	8.315	9.699	11.386
9	0.387420	1.457	2.898	4.324	5.787	7.242	8.657	9.839	11.182
10	0.348678	1.570	3.156	4.739	6.348	8.127	9.827	11.698	13.910
11	0.313811	1.600	3.171	4.688	6.168	7.542	8.842	10.534	11.980
12	0.282430	1.618	3.245	4.864	6.306	7.683	9.082	11.810	15.275
13	0.254187	1.680	3.405	5.111	6.710	8.172	10.369	18.193	17.005
14	0.228768	1.709	3.403	5.048	6.564	7.620	10.601	670.898	655.622
15	0.205891	1.780	3.567	5.412	7.290	8.696	10.427	0	0
16	0.185302	1.822	3.601	5.370	8.635	12.082	17.006	0	0
17	0.166772	1.843	3.657	5.354	6.277	6.578	0	0	0
18	0.150095	1.866	3.488	4.986	8.921	674.108	655.638	0	0
19	0.135085	1.885	3.621	4.996	6.206	0	0	0	0

Note: Standard dev. = σ_r = 0.2845; $1/2\sigma_r$ = 0.14225; $2\sigma_r$ = 0.569

c) AR(1) residuals

j	ε_j	h=2	h=4	h=6	h=8	h=10	h=12	h=14	h=16
18	0.150095	0.438	0.792	1.093	1.364	1.603	1.799	1.986	2.163
19	0.135085	0.557	1.013	1.390	1.718	1.997	2.221	2.433	2.626
20	0.121577	0.667	1.210	1.657	2.039	2.361	2.641	2.910	3.150
21	0.109419	0.782	1.422	1.957	2.413	2.792	3.115	3.435	3.721
22	0.098477	0.896	1.619	2.214	2.698	3.090	3.436	3.768	4.078
23	0.088629	1.019	1.847	2.520	3.055	3.501	3.920	4.313	4.699
24	0.079766	1.126	2.046	2.787	3.365	3.884	4.372	4.857	5.329
25	0.071790	1.237	2.238	3.057	3.727	4.311	4.851	5.327	5.775
26	0.064611	1.319	2.366	3.253	3.992	4.669	5.333	5.944	6.561
27	0.058150	1.416	2.544	3.510	4.292	5.050	5.764	6.517	7.122
28	0.052335	1.493	2.676	3.698	4.519	5.260	5.913	6.440	6.931
29	0.047101	1.577	2.905	4.088	5.115	6.059	6.974	7.650	8.256
30	0.042391	1.656	3.094	4.337	5.360	6.132	6.471	6.783	7.127
31	0.038152	1.699	3.170	4.327	5.168	6.102	7.124	7.981	8.500
32	0.034337	1.746	3.301	4.812	6.045	7.494	9.629	11.739	14.018
33	0.030903	1.792	3.423	4.953	6.441	8.704	11.261	14.274	25.378

Note: Standard dev. = σ_r = 0.0676; $1/2\sigma_r$ = 0.0338; $2\sigma_r$ = 0.1352. C_h is computed for values of ε in the range $[1/2\sigma_r, 2\sigma_r]$

TABLE 10

Correlation Dimension for 20 bootstrapped series of AR(1) Residuals

a) Estimated maxima and minima

	h=2		h=4		h=6		h=8	
j	max	min	max	min	max	min	max	min
18	0.97	0.43	1.93	0.86	2.91	1.31	3.96	1.75
19	1.13	0.53	2.25	1.05	3.40	1.59	4.61	2.14
20	1.21	0.62	2.43	1.24	3.68	1.87	5.03	2.52
21	1.31	0.74	2.60	1.47	3.94	2.22	5.36	2.97
22	1.41	0.86	2.80	1.72	4.16	2.60	5.62	3.51
23	1.50	1.01	3.01	2.01	4.46	3.02	5.98	4.03
24	1.58	1.11	3.14	2.22	4.68	3.32	6.28	4.45
25	1.65	1.23	3.29	2.46	4.95	3.72	6.28	4.93
26	1.73	1.35	3.44	2.72	4.99	4.03	6.56	5.36
27	1.75	1.44	3.51	2.87	5.21	4.24	6.99	5.50
28	1.79	1.53	3.67	3.04	5.67	4.53	8.31	6.11
29	1.84	1.60	3.66	3.20	5.50	4.62	7.51	5.86
30	1.90	1.68	3.86	3.36	5.57	4.94	7.41	6.33
31	1.90	1.71	3.89	3.38	5.84	4.99	8.28	6.09
32	1.90	1.75	3.82	3.50	6.61	4.87	8.69	5.03
33	1.90	1.78	3.92	3.52	7.75	4.95	11.48	5.35

a) Estimated maxima and minima

	h=10		h=12		h=14		h=16	
j	max	min	max	min	max	min	max	min
18	5.03	2.18	6.11	2.58	7.25	2.97	8.36	3.35
19	5.87	2.67	7.19	3.18	8.69	3.66	10.07	4.16
20	6.33	3.13	7.78	3.71	9.42	4.26	10.95	4.83
21	6.78	3.70	8.26	4.38	9.66	5.04	10.46	5.73
22	7.24	4.40	8.92	5.23	10.90	6.07	13.12	6.94
23	7.50	5.02	9.50	5.94	13.19	6.78	24.58	7.60
24	7.73	5.53	9.01	6.47	11.26	7.33	14.25	8.17
25	8.43	6.03	11.1	7.11	14.67	8.16	18.69	6.57
26	8.59	6.50	13.01	7.52	27.43	8.54	672.63	9.27
27	9.04	6.69	13.35	7.64	662.21	7.52	18.59	0
28	9.08	7.09	11.89	5.92	16.55	0	688.81	0
29	10.30	6.75	16.05	0	22.75	0	677.41	0
30	10.83	2.73	13.15	0	666.07	0	668.80	0
31	16.18	3.84	668.8	0	672.65	0	655.64	0
32	662.24	2.73	668.82	0	668.82	0	655.66	0
33	668.72	0	655.57	0	662.14	0	6.57	0

b) Estimated means

j	h=2	h=4	h=6	h=8	h=10	h=12	h=14	h=16
18	0.54	1.08	1.62	2.15	2.70	3.24	3.79	4.34
19	0.67	1.34	1.99	2.66	3.33	4.00	4.69	5.38
20	0.78	1.56	2.33	3.10	3.87	4.64	5.42	6.20
21	0.90	1.79	2.68	3.58	4.48	5.37	6.29	7.18
22	1.01	2.02	3.01	4.01	5.03	6.02	7.05	8.08
23	1.14	2.29	3.41	4.52	5.63	6.74	7.93	9.50
24	1.24	2.49	3.71	4.94	6.14	7.29	8.45	9.65
25	1.36	2.72	4.08	5.43	6.76	8.10	9.58	11.04
26	1.44	2.90	4.32	5.75	7.28	8.97	11.48	0
27	1.53	3.07	4.61	6.10	7.59	9.15	0	0
28	1.62	3.25	4.92	6.63	8.16	9.89	0	0
29	1.68	3.37	5.02	6.64	8.18	0	0	0
30	1.74	3.48	5.21	6.93	8.16	0	0	0
31	1.76	3.54	5.30	7.03	8.57	0	0	0
32	1.81	3.64	5.52	7.07	0	0	0	0
33	1.84	3.68	5.66	0	0	0	0	0

Note: Maxima. minima and mean values of C_h for 20 bootstrapped series. C_h is calculated for values of ε in the range $[1/2\sigma_r, 2\sigma_r]$, where σ_r is the standard deviation of the original series. $\sigma_r = 0.0676$; $1/2\sigma_r = 0.0338$; $2\sigma_r = 0.1352$.

b) Hsied third order moments test

The second approach used to verify the existence of chaos will be based on the third order moments of stock returns. As we have seen (Table 4) there is a significant evidence of nonlinearity in the stock returns. Unfortunately, both additive and multiplicative dependence imply that x_t^2 is correlated with its own lags, so that evidence of nonlinearity does not offer any indication of the type of dependence and the ways to model it. In the case of additive dependence we will have a model *non-linear in mean* of the type:

$$x_t = f(x_{t-1}, x_{t-2}, \ldots, x_{t-k}) + \varepsilon_t \qquad [3]$$

that in the case of multiplicative dependence turns to a model *non-linear in variance:*

$$x_t = g(x_{t-1}, x_{t-2}, \ldots, x_{t-k})\varepsilon_t \qquad [4]$$

where f and g are nonlinear functions and $E[\varepsilon_t \mid x_{t-1}, x_{t-2}, \ldots, x_{t-k}] = 0$.

Note that in [3] the nonlinearity only enters trough the mean of the process while in [4] it only does through its variance. Obviously, a (purely) chaotic process is a particular type of model non-linear in mean where $\varepsilon_t = 0$ for all t. Hsieh (1989) has proposed a simple test to discriminate both types of dependence. To test the hypothesis that $f = 0$ against $f \neq 0$, Hsieh suggests using the third order moments $\rho(i,j) = E[x_t x_{t-i} x_{t-j}] / V(x_t)^{3/2} (i,j>0)$ that, under the null, are equal to zero. Thus, one can estimate these moments by the sample ones $r(i,j) = [\sum x_t x_{t-i} x_{t-j} / T] / [\sum x_t^2 / T]^{3/2}$ and consider that, under the null, $r(i,j)$ is asymptotically distributed as a $N(0, \omega(i,j)/\sigma^6)$ where $\omega(i,j)/\sigma^6$ can be consistently estimated by $[(1/T) \sum x_t^2 x_{t-i}^2 x_{t-j}^2] / [(1/T) \sum x_t^2]^3$. This test is designed so that it will not reject models non-linear in variance.

TABLE 11
Bicorrelation coefficients and Hsieh statistic for testing nonlinearity in mean

i	j	$\rho(i,j)$	t_ρ	i	j	$\rho(i,j)$	t_ρ
1	1	0.408124	1.239571	4	3	0.036995	0.734385
2	1	−0.01281	−0.09041	4	4	0.076793	0.749471
2	2	0.032200	0.140924	5	1	−0.05827	−1.18813
3	1	0.118371	1.845363	5	2	−0.07663	−1.31431
3	2	−0.04081	−0.64335	5	3	0.008629	0.110033
3	3	0.086481	0.917189	5	4	−0.00555	−0.07088
4	1	0.014627	0.293484	5	5	0.004856	0.032129
4	2	0.349322	0.349322				

Note: Bicorrelation coefficients and Hsieh third order moments test for the standardized residuals of the AR(1) filter: $z_t = (x_t - \mu)/\sigma$. $t_\rho = T^{1/2}\, r(i,j) / [\omega\,(i,j)/\sigma^6]^{1/2} \sim_{H0} N(0,1)$. Critical values: 1.645 (10%); 1.960 (5%); 2.576 (1%).

In Table 11 we have calculated the third order moments for the standardized residuals of the AR(1), none of them is significantly different from zero, which implies that there is no evidence in favor of nonlinearity in mean (and, consequently, of deterministic chaos). Unfortunately, though a conditional mean equal to zero implies that third order moments are zero too, the inverse is not true (Pemberton and Tong, 1981) so that nonlinearity in mean cannot be properly rejected.

c) Nonparametric forecasts

We will now employ a «direct» method to capture the natures of the conditional mean that consists in estimating it nonparametrically. The catalog of tecniques useful for estimating nonparametrically f in [3] is huge: flexible Fourier forms, locally weighted regression, MARS, kernels, splines, etc. We will use Artificial Neural Networks (ANN's), a less well known and biologically inspired technique which has been proved quite useful in several domains. There are many different ANN's models useful in an econometric context (see Kuan and White, 1994 for an excellent review) but in this work we will only use single hidden layer feedforward networks. Due to the scope of the paper we will simplify greatly our exposition, a clasical reference is Rumelhart *et al.* (1986).

A single hidden layer feedforward network with r inputs, q hidden units and one output is a parametrization of the form:

$$f^*(x) = \sum(q) = x'\theta + \beta_0 + \sum_{j=1}^{q} \beta_j\, G(x_a' \gamma_j) \qquad [5]$$

where $x_a' = (1, x_1, x_2, ..., x_r)$, $\gamma_j = (\gamma_{0j}, \gamma_{1j}, ..., \gamma_{ij}, ..., \gamma_{rj}) \in \mathbb{R}^{r+1}$, $\beta = (\beta_0, \beta_1, ..., \beta_q) \in \mathbb{R}^{q+1}$, $\theta = (\theta_1, \theta_2, ..., \theta_r) \in \mathbb{R}^r$ and G the logistic function $G(x) = (1+e^{-x})^{-1}$.

Observe that the first two terms in [5] are just a linear regression on the explanatory variables $x_1, x_2, ..., x_r$ and an intercept β_0. In our times series context the explanatory variables will be the values of the series lagged r periods so that we have $x = (x_{t-r}, x_{t-r+1}, ..., x_{t-1})$, and we can think of these two first terms as an AR(r) model.

The third part is a composition of sigmoidal functions of x that will try to capture the nonlinearities (if any) of the DGP. It can be demonstrated (see, for example, Hornik *et al.*, 1989) that a $\sum(q)$ network can approximate arbitrarily well any Borel measurable function, provided that q is sufficiently large. So, we can employ $\sum(q)$ to approximate f in [3].

Though flexible, these models are parametrically intensive (for any q we will have $(r+1)+(r+2)q$ parameters) and two problems arise (see Olmeda, 1993, and references therein). The first one is related to the estimation of

[5] by least squares. Since a gradient-based method would lead to a local minimum it is necessary to use global optimization methods or to restart the search with a different set of initial conditions, here we will employ the latter procedure. The second problem comes from overfitting the data used to construct the model («training set») resulting in large prediction errors over the «testing set». To avoid overfitting one can addopt different strategies but in this paper we use a simple approach constructing several models with a different number of parameters. It is important to note that if one wants to make some inference from the models built, these problems (as well as some others rather technical ones) should be alleviated.

To see if the AR(1) residuals are generated by a chaotic process, we have built four different $\sum(q)$ ANN's models like [5] with 6 inputs and 6, 12, 16 and 30 hidden units. This number of hidden units should be enough to approximate any low dimensional chaotic process, since other well-known deterministic chaotic processes as the logistic or McKey-Glass equations can be modelled with as few as 5 hidden units and a few hundred of observations. As the linear structure of the data has been previously removed the θ terms should not be included. We run a gradient-based optimization method («backpropagation») based on the stochastic approximation method of Robbins and Monro (1951) (for additional details refer to White, 1989). We used five different sets of parameters (step size, momentum, etc.) and initial estimates of γ and β. The sample used for modelling consisted in the first 1000 observations, keeping the last 255 ones to test the forecasting accuracy.

In Table 12 (upper part), we have computed the forecasting performance of the ANN's models relative to the one of the random walk by computing the ratio of their Root Mean Squared Error (RMSE). A value higher than 1 indicates poorer performance than the random walk. As one can see, the forecasting accuracy of ANN's models and the random walk is the same, both in-sample and out-of-sample. The best ANN's model shows an improvement of less than 0.2% over the random walk in-sample and less than 2% out-of-sample.

One could argue, though, that the estimated dimension could be higher than 6, so that dependence could occur at the, say, seventh lag. To avoid this critique, we have built another set of ANN's models with 8, 12, 16 and 30 hidden units that include as explanatory variables the lagged ones up to the twelfth order. Appart from the computational burdensome, one can always use this approach of using more explanatory variables than needed as the ANN's models (contrary to some other nonparametric approaches such as nearest neighbours) will ignore the irrelevant ones. From the lower part of Table 12 we see that increasing the order of the lags does not improve the forecasting accuracy, the best ANN's model is around 0.4% better than the random walk in-sample and around 1.5% out-of-sample.

TABLE 12
Forecast errors
Neural Nets versus Random Walk

Lags = 6							
training set				testing set			
q=6	q=12	q=16	q=30	q=6	q=12	q=16	q=30
1.017468	1.002319	0.999208*	1.008429	1.066184	0.989912*	0.999989*	0.986778*
1.016834	1.005144	0.999619*	0.999624*	1.064807	0.987524*	0.999810*	0.999395*
1.018432	0.999837*	1.000491	0.998834*	1.068555	0.997435*	0.994168*	1.002007
1.017570	0.999576*	1.005312	1.009105	1.066539	1.001540	0.987447*	1.043835
1.009816	1.000551	0.999629*	1.020362	0.986385*	0.993924*	0.998415*	1.072792

Lags = 12							
training set				testing set			
q=8	q=12	q=16	q=30	q=8	q=12	q=16	q=30
1.019251	0.998876*	0.999316*	0.996765*	1.067677	1.004473	1.014130	1.007818
0.998774	1.006401	1.003957	0.996074*	0.996653*	1.037157	0.985901*	1.002192
0.998276*	1.001132	1.005963	0.997591*	0.996340*	1.019489	0.986049*	1.000808
0.999765*	0.998662*	0.997762*	1.005572	0.993798*	1.006181	0.997175*	0.985853*
1.099800	0.998603*	0.998195*	1.006909	0.993494*	1.006122	0.996857*	1.014185

Note: Ratio of the Root Mean Squared Error (RMSE) of forecasts from a $\Sigma(q)$ single hidden layer neural net (with q hidden units) versus a random walk (zero drift) model. A value of RMSE ($\Sigma(q)$)/RMSE (random walk) smaller than one denotes better performance than the random walk. Each row is a different $\Sigma(q)$ net.

It can be possible, though, that even if we are unable to improve the forecasts of a random walk under a RMSE criterion we could predict more accurately the sign of returns. To see if this is the case we have computed the number of directions correctly predicted by the networks and divided it by the number of predictions. A value higher than 0.5 indicates a performance better than a random directional forecast. As we can see from Table 13 (upper part) there are many models $\sum(q)$ with six inputs that outperform the random forecasts but only five of them show a statistically significant improvement over the random forecasts out-of-sample. This number increases to 6 for the models that consider lagged variables up to the twelfth order (lower part of Table 13). These results seem to indicate that though the ANN's are not superior under an RMSE criterion they are better predicting the sign of the change. As the proportion of ups and downs is different to one and also different between the training and testing sets these results could be biased. To avoid this problem we have built an artificial decisor that follows a naïve strategy that consists in predicting a sign change equal to the previously observed, in this way we partially incorporate the unequal proportion of signs. From Table 14 we see that the same ANN's models show a performance that is better than the naïve predictor (around 6% in the best case).

TABLE 13

Directional Forecasts

Neural Nets

Lags = 6

training set				testing set			
q=6	q=12	q=16	q=30	q=6	q=12	q=16	q=30
0.522132	0.484909	0.523138	0.477867	0.427451	0.572549*	0.533333	0.572549*
0.522132	0.471831	0.526157	0.531187	0.427451	0.572549*	0.525490	0.552941
0.522132	0.503018	0.480885	0.500000	0.427451	0.541176	0.560784	0.486274
0.522132	0.523138	0.471831	0.522132	0.427451	0.501960	0.572549*	0.427451
0.477867	0.479879	0.513078	0.522132	0.572549*	0.560784	0.560784	0.427451

Lags = 12

training set				testing set			
q=8	q=12	q=16	q=30	q=8	q=12	q=16	q=30
0.524291	0.530364	0.522267	0.512145	0.417451	0.470588	0.431372	0.470588
0.513157	0.524291	0.473684	0.524291	0.560784	0.427451	0.568627*	0.501960
0.511133	0.525303	0.474696	0.526315	0.568625*	0.432529	0.572549*	0.521568
0.500000	0.522267	0.511133	0.478745	0.576470*	0.466666	0.556862	0.572549*
0.507085	0.524291	0.514170	0.475708	0.568627*	0.462745	0.560784	0.435294

Note: Number of correct predicted directions from a $\Sigma(q)$ neural net/number of predictions. A value bigger than 0.5 denotes better performance than a random directional forecast. The normalized standard error is 0.031.
(*) Significant at the 5% level.

TABLE 14

Directional Forecasts

Neural Nets versus Naïve Strategy

Lags = 6

training set				testing set			
q=6	q=12	q=16	q=30	q=6	q=12	q=16	q=30
1.067437	1.149378	1.065385	1.166316	1.255881	0.938356*	1.007353	0.938356*
1.067437	1.181237	1.059273	1.049242	1.256881	0.938356*	1.022388	0.971631*
1.067437	1.108000	1.158996	1.114688	1.256881	0.992753*	0.958042*	1.104839
1.067437	1.065385	1.181237	1.067437	1.256881	1.070313	0.938356*	1.256881
1.166316	1.161426	1.086275	1.067437	0.938356*	0.958042*	0.958042*	1.256881

Lags = 12

training set				testing set			
q=8	q=12	q=16	q=30	q=8	q=12	q=16	q=30
1.063707	1.051527	1.067829	1.088933	1.256881	1.141667	1.245455	1.141667
1.086785	1.063707	1.177350	1.063707	0.958042*	1.256881	0.944827*	1.070313
1.091089	1.061657	1.174840	1.059615	0.944827*	1.268519	0.938356*	1.030075
1.115385	1.067829	1.091089	1.164905	0.931972*	1.151126	0.964788*	0.938356*
1.099800	1.063707	1.084646	1.172340	0.944827*	1.161017	0.958042*	1.234234

Note: Number of correct predicted directions from a naïve strategy/number of correct predicted directions from a $\Sigma(q)$ neural net. A value smaller than one denotes better performance.

These results not only show (again) that the residuals are not iid but they also suggest that there is some non-linear dependence in mean between them. It also seems that this kind of dependence is so weak that it cannot be exploited to improve point forecasts so that one should consider using a different information set (including a volatility index, for example) to account for the nonlinearity in variance.

Finally, we have to note that the method we have employed for training the models (backpropagation) asumes a RMSE criterion for testing (and a quadratic loss function of the decisor). This means that under the criterion of the number of correct direction forecasts the models built are possibly suboptimal and that using a different criterion for training (negative entropy, for example) sign forecasts could be possibly improved.

d) Nonlinearity in variance

Our earlier analysis showed that a pure chaotic explanation is unplausible and we also conjectured that a stochastic process non-linear in mean cannot account for the rejection of the iid hypothesis by the BDS test. Since the BDS test has also power against alternatives such as ARCH (Brock *et al.*, 1991), it can be possible that the rejection could be attributable to heteroskedasticity. Perhaps the two best known models of nonlinearity in variance are the ARCH(p) model of Engle (Engle, 1982) and its generalization, the GARCH(p,q) (Bollerslew, 1986).

To verify if simple models such as ARCH(p) or GARCH(p,q) can account for the nonlinearity found, we fitted ARCH(p) ($p \leq 5$) and GARCH(p,q) ($p \leq 3$, $q \leq 5$) to the AR(1) residuals. We also considered including the conditional variance as a regressor in the mean equation (ARCH-M and GARCH-M models, respectively) but it did not proved helpful. Since ARCH(p) and GARCH(p,q) are nested models we employed a likelihood-ratio test to choose among them, finding the GARCH(1,1) as the best.

In Table 15 we present the results. As one can see, the GARCH(1,1) model does not provide a completely satisfactory explanation of the data. The standardized residuals are not normally distributed, suggesting that the model fitted could be considerably improved by incorporating additional variables in the mean (such as interest rates, as in Baillie and DeGenaro, 1990), using a different conditional density to account for the leptokurtosis (Baillie and Bollerslev, 1989) or employing derivates such as the Exponential ARCH (Nelson, 1991) or the GARCH Exponential AR (LeBaron, 1992). Nevertheless, the model is able to capture any linear and non-linear dependence in the squared residuals, at the light of the Ljung-Box and the McLeod-Li tests.

TABLE 15
GARCH (1,1) model for the AR(1) residuals
$x_t = \sigma_t u_t$, $\sigma_t^2 = \alpha_0 + \alpha_1 x_{t-1}^2 + \beta\sigma_{t-1}^2$, $u_t \sim N(0,1)$.

α_0	α_1	β	log likelihood	$S(e_t/\sigma_t)$	$K(e_t/\sigma_t)$	Ljung-Box	Mcleod-Li
.815*10^{-5}	.100	.819	4091	−1.46	20.62	13.98	1.16
(6.52)	(7.19)	(35.18)		(0.0691)	(0.1382)		

Note: Estimated parameters of the GARCH(1,1) model (t-statistics in parentheses). S and K are the Skewness and Kurtosis coefficients for the standardized residuals. The Ljung-Box statistic is computed for the standardized residuals and the McLeod-Li for the squared residuals (number of lags equal 12). The value of x_{12}^2 corresponding to a 5% level test is 21.

In Table 16 we show the results obtained with the BDS test. When comparing the results with the ones of Table 5 one can notice the decrease in the values of the BDS. This suggest that the GARCH model accounted partially for the nonlinearity found. A problem with the BDS test when applied to the residuals of an ARCH-type model is that the asymptotic distribution of the BDS test for standardized residuals is not $N(0,1)$, so that one cannot use the typical critical values. Alternatively, one can use the simulated critical values provided in Brock *et al.* (1991, pp. 276-279) as a guideline. By doing this we find that the BDS values are still significant at the 5% level, showing that the GARCH(1,1) model could not account for all the nonlinearity.

TABLE 16
BDS Statistic for standardized GARCH(1,1) residuals

h	$1/2\sigma$	σ	$3/2\sigma$	2σ
2	4.18	3.07	2.22	2.12
3	5.06	3.56	2.49	2.34
4	6.45	4.30	2.84	2.37
5	6.97	4.52	2.76	2.09
6	9.07	5.57	3.30	2.33
7	12.49	6.46	3.62	2.44
8	16.75	7.49	3.93	2.56

Note: BDS statistic for standardized residuals $s_t = e_t/\sigma_t$ of the model $x_t = \sigma_t u_t$, $\sigma_t^2 = \alpha_0 + \alpha_1 x_{t-1}^2 + \beta\sigma_{t-1}^2$, $u_t \sim N(0,1)$. Estimated parameters (t-statistics in parentheses): $\alpha_0 = .815{*}10^{-5}$ (6,52, $\alpha_1 = .100$ (7.19), $\beta = .819$ (35.18). Critical values: 1.645 (10%); 1.960 (5%); 2.326 (2%); 2.576 (1%). $\sigma(s_t) = \sigma = 1.041$.

4. Conclusion and future work

In this paper we have presented some preliminary results that show that the hypothesis of iid returns in the Spanish stock market cannot be mantained. Using the BDS statistic we concluded that after removing the linear dependence in the stock returns series the hypothesis of iid returns is again rejected, so the series is either non-stationary or the observations depend in a non-linear fashion. Using different sampling frequencies we determined that the iid hypothesis can be also discarded. In the case of high frequency data the temporal horizon is so short that a changing DGP seems not plausible so that nonstationarity does not seem to be the cause of rejection and the only alternative is that of non-linear dependence.

We have also tried to determine the nature of the non-linear dependence employing three different approaches. The estimates of the dimension for the original series and the resampled ones are approximately the same for low embedding dimensions suggesting a similar level of stochasticity. While the results are inconclusive due to the sample size, the hypothesis of deterministic low dimensional chaos (a purely deterministic non-linear in mean process) can not be maintained. If chaos is present it must be of a dimension so high as to make it appear as random. We have also found that the dimension estimated are not robust to the size of the sample used. If one is trying to properly verify or discard a chaotic DGP using dimension estimates, one needs to follow some (unknown) «rules of thumb» between sample size, noise level, ε values and embedding dimension, so as to decide if the problem is, simply, intractable.

Using the Hsieh statistic we were also unable to conclude nonlinearity in mean of the DGP. Unfortunately this test can only detect some types of this kind of nonlinearity so that it does not provide conclusive results.

Our last approach was to consider that if the DGP was truly chaotic the forecasts could be dramatically improved. We did not succeed in this task. Even though we used a powerful nonparametric technique and many different models we could only beat a random forecast predicting the direction of change slightly better.

Finally, we studied if simple ARCH-type models could account for the detected nonlinearity. We found that though the estimated model reduces the values of the BDS statistic they are still statistically significant and, moreover, the residuals of this model are not normally distributed.

Having found a strong non-linear dependence, these results suggest that returns only show a weak dependence nonlinear in mean and that most of the dependence occurs through variance, possibly in a non-linear fashion. We conjecture that nonparametric models that include an index of volatility can provide superior forecasts to the ones obtained here.

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Resumen

Este trabajo explora la existencia de dinámica no lineal y caos en el mercado de valores español. Usando el contraste BDS, se observa que las series de rendimientos muestran dependencias significativas lineales y no lineales. También se observa que las estimaciones de dimensión son parecidas a las de procesos con cierto nivel de ruido, por lo que no puede deducirse la existencia de caos de baja dimensión. Se muestra que la dependencia en media es débil y que las predicciones basadas en este tipo de información no mejoran las realizadas a partir de la hipótesis de paseo aleatorio. Finalmente, se demuestra que los modelos GARCH no explican toda la no linealidad encontrada.

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