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MEASURING THE DEGREE OF FULFILLMENT OF THE LAW OF ONE PRICE. APPLICATIONS TO FINANCIAL MARKET INTEGRATION

ALEJANDRO BALBÁS
MARÍA JOSÉ MUÑOZ
Universidad Carlos III de Madrid

In this paper we introduce two measures of the degree of fulfillment ot the Law of One Price. Since these measures are defined by means of non-differentiable optimization programs, we reformulate the problems in such a way that the measures are characterized by means of saddle point conditions. Many empirical papers anlyze well-known arbitrage strategies. The measures introduced here follow a different approach, because they globally focus on the market to find its arbitrage opportunities without studying special strategies. Our approach is useful to study the integration of different financial markets and also, could be applied to markets with frictions. We relate our measures with previous measures in the literature and we show that, with our approach, the inconsistency of the market with the Law of One Price is reflected in monetary terms. The theory developed below may be easily extended to reflect and analyze the level of arbitrage opportunities in monetary terms. (JEL C23)

1. Introduction

The fulfillment of the Law of One Price (LOP) or the absence of arbitrage opportunities in financial markets may be characterized by the existence of state prices with appropriate properties (Ingersoll (1987) or Chamberlain and Rothschild(1983)). Chen and Knez (1995) characterize the absence of arbitrage opportunities among different financial markets by means of the vanishing of a certain integration measure. In their empirical analysis of the markets NYSE and NASDAQ , they compute that, based on the data, the two markets seem to violate the strong-form integration (i.e., cross-market arbitrage may be possible).

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They also check that the measure depends crucially on the frictionless market assumption and consequently they suggest the convenience of extending their discussion to economies with trading frictions and transaction costs.

When the LOP is characterized by the existence of state prices, we may only know whether the LOP holds or does not hold. We follow Chen and Knez(1995)'s idea of introducing a measure to characterize the fulfillment of the LOP. This represented a breakthrough in quantifying pricing discrepancy between markets. The Chen and Knez measure and the functions we propose reflect different things and hence they serve different purposes (though they are related as we show). Our measures are proposed as a tool for measuring the extent of deviation from the LOP, and hence it is not necessary to deal with more than one market. Consequently, the present paper analyzes first the level of fulfillment of the LOP in a single financial market. When a market violates the LOP, prices must be out of line with each other so that one can find a replicating portfolio for a given payoff with arbitrary price. Therefore, we must look for relative measures if we are interested in measuring the level of fulfillment of the LOP in monetary terms. Thus, assuming that the traded assets all have positive prices we were interested in the minimum feasible cost among the replicating portfolios for the zero payoff with the same total exchanged (sold and purchased) assets price. Thus we compute how much money an investor can win from an arbitrage portfolio with respect to the total price of all the exchanged assets. When we compute this maximum relative profit (hereafter denoted by l), it will be possible to take into account the transacion costs, discount them and guarantee if there still are arbitrage opportunities. Hence, the theory we develop here could be considered an alternative to the results of Prisman(1986) for a market with frictions and it is very related to the results of Jouini and Kallal (1995).

Many empirical papers test the efficiency of financial markets, or even some well-known arbitrage strategies, usually concluding that these strategies could be implemented if the transaction costs were small enough, (Hudson, Dempsey and Keasey (1996) or Kamara and Miller (1995)). The measure we introduce below analyzes the market globally, since we are looking for all the possible ways of replicating portfolios to obtain monetary profits, and therefore it does not only take into account specific strategies.

As already said, the measure we propose gives the quotient between the total amount of money an agent can win and the price of the assets he/she has to exchange to replicate his/her portfolios. From a mathematical point of view, this measure is obtained from solving an optimization program. However, it is not easy to solve this problem since some of the constraints are given by strict inequalities. Hence, the existence of a solution is not guaranteed. Furthermore, even assuming the existence of a solution, the usual analytic techniques do not apply, since the objective is a non-differentiable function.

To avoid these difficulties, we organize the paper as follows. The second section is devoted to present the basic assumptions, the notations and some classic and important results which will frequently be applied. A measure m of the degree of fulfillment of the LOP is introduced in the third section. The consistency of m will allow us to guarantee in Section 5 the consistency of l. To do it, we begin by assuming that each agent holds a portfolio given by a vector $h = (h_1, h_2, \cdots h_n)$ (which depends on the agent) such that $h_i \geq 0$, where h_i is the number of units on the i-th asset. We also assume that the agents cannot replicate their portfolios by selling more than h_i units in each asset i (an agent cannot sell what he/she does not have). By solving an optimization program we introduce the function $\phi(h)$, the minimum feasible cost among the replicating portfolios for the zero payoff, when the short positions to be taken in the traded assets are bounded from below by h. ϕ is identically zero if and only if the LOP holds. The fact that ϕ is an homogeneous function of degree one leads to maximize ϕ among all the priced one portfolios h and it reflects the maximum profit an investor can obtain by replicating the portfolios h relative to the price of h. This optimal value , which again vanishes if and only if the LOP holds, will be our measure m of the degree of fulfillment of the LOP. Finally, m is easy to implement in practical situations. A simple linear program leads to m, the portfolio h^* where ϕ maximizes, and the optimal arbitrage portfolio x^*.

The fourth section is devoted to prove a Saddle Point Theorem (Theorem 2). We also show some intuitive interpretations of the Saddle Point Theorem.

Short-selling restrictions disappear in the fifth section: we allow the investors to hold initial portfolios h with short positions in each asset. We also assume there is no limit on the short-selling restrictions that can be taken to replicate portfolios. Under these assumptions, we prove Theorem 3, one of the most important results of the present paper, since it guarantees that after absolutely relaxing the restrictions imposed to the short positions, we obtain the same value for the level of fulfillment (or violation) of the LOP. Hence, m is also the optimal value of another program, where the objective is the quotient between the profit an agent can win by replicating (without restrictions) his/her arbitrary portfolio and the price of the sold assets. Furthermore, the best way to replicate is still given by the portfolio x^* already computed. If we are interested in measuring the profit relative to the price of the exchanged assets, Theorem 4 shows that x^* is again the best way to replicate any portfolio. In fact, it is enough to substitute m by $l = \frac{m}{2-m}$ to obtain the optimal value of the relation we are interested in. As we show in this section, both measures m and l have similar properties.

Note that the ideas in the fifth section have important consequences, which are far from evident: although in Section 3 we imposed short-selling restrictions to avoid mathematical difficulties, our results in Section 5 show that the measures m and l do not depend on these restrictions. From the most constrained conditions (no initial short positions are permitted, and the agents cannot sell what they do not have) to the most relaxed ones (no limit on the short positions of the initial portfolio h to be replicated, and no limit on the number of assets to be sold) we obtain the same measure m and the same optimal portfolio x^* to implement the arbitrage. In the first case not all the investors can win the maximum relative profits given by m (an agent needs an initial portfolio proportional to h^*), while in the second case this level of maximum relative profit is available to any agent. Of course, we would obtain the same values for m, l and x^* if we worked under assumptions not so restrictive or not so relaxed.

These results could also be interesting in optimization theory. In fact, a saddle point condition in a linear optimization program yields a solution for a non-differentiable one.

In the sixth section, we measure the degree of integration of two or more financial markets. To do it, we work in a global market which contains all the assets of the different ones, and we compute m on this global market as a measure of their integration. This represents an alternative to the results in Chen and Knez(1995), where they measure how the markets jointly verify this law by computing the distances among the families of state prices of each market. However, in our work we do not need to impose the assumption that the LOP holds separately on each market. In this case, the estimation of m in the combined market gives no indication as to wether the violation of the LOP comes from market A, market B, or cross-market violations and then it is convenient to estimate separately m on each market to find out the relative arbitrage profit due to cross-market violations.

Theorem 5 shows that our measure is continuous with respect to the Chen and Knez(1995) measure (hereafter denoted by g). Therefore, controlling the latter we also control the first. The opposite is false in general: for instance, m can take small values while g remains constant. Two simple examples illustrate this fact and show that g is not always sensitive to the profits the investors can win due to violation of the LOP across the two markets. This is an important fact to realize the differences between g and m: g reflects asset pricing discrepancy between markets while m and l measure relative profits available by the agents in the combined market.

The first example also illustrates that the way in which each market is divided into sub-markets has a strong effect on the final value of g, and then g can not be applied if one of the two markets violates the LOP. Finally, both examples illustrate the fact that m (and consequently l) is continuous with respect the prices and the payoffs of the traded assets. This is not the case for g. Such a discontinuity makes g very sensitive with respect to eventual errors committed in the data estimation process.

Chen and Knez (1995) also define what they call a strong integration measure, based on the notion that two markets cannot be integrated in a stronger sense if there are cross-market arbitrage opportunities. They suppose two markets such that there is no arbitrage opportunity on either market. Then, they define a measure of the extent of representation for the two markets as the minimum mean-square distance between the respective sets of nonnegative stochastic discount factors. The theory here developed may also be easily extended to reflect the level of arbitrage opportunities in the strong form or cross-market arbitrage opportunities when considering strong integration for two or more markets.

The last section summarizes the most important conclusions of this paper and the appendix presents the proofs of the main results. All our results are focused on studying the degree of fulfillment (or violation) of the LOP in a financial market or among different ones, but, as already said, an analogous analysis may be extended to measure the level of arbitrage opportunities (in the strong form). To do this, we must look for maximum relative profits among the portfolios whose payoffs are positive (instead of zero payoffs). Proceeding in the same way, it is possible to define two functions reflecting the level of arbitrage opportunities in monetary terms. We only have to replace the equality constraint by an inequality in the optimization problem leading to define the function $\phi(h)$. We can similarly define two measures taking into account this inequality constraint and derive the corresponding results.

2. Preliminaries

Consider an economy endowed with a Hausdorff compact topological space K, on which the linear space $C(K)$ of all continuous functions over $\mathbb{R}$ is defined. When equipped with the norm $\|\alpha\| = \sup\{|\alpha(k)| \mid k \in K\}$ for any $\alpha \in C(K)$, the space $M(K)$ of Radon measures over K is known to be the dual space of $C(K)$ (Riesz representation Theorem). Here we are assuming that K is the set of outcome states and for some $\alpha \in C(K)$, $\alpha(k)$ represents the payoff of a portfolio in the state of nature k for every $k \in K$. This restriction to continuous contingent claims is made for expositional and mathematical ease. In many papers (Harrison and Kreps(1979), Chamberlain amd Rothschild(1983),Chen and Knez (1995)$\cdots$), Hilbert space methods are used to represent pricing functions and to characterize the absence of arbitrage across a market. Thus, much of their analysis leads them to consider an economy endowed with a probability space on which the space of all square-integrable functions is defined. In dealing with a finite number of states of nature both models coincide with the classical theory, and in most other cases it is possible to deduce one model from the other.

Let the number of assets be finite and indexed by $\{1, \cdots n\}$. The current prices of the assets are $p = (p_1, p_2, \cdots, p_n)$, and the total payoff on the i^{th} asset is $\alpha_i \in C(K)$. We denote such a market by $M_{p,\alpha}$ where $p = (p_1, p_2, \cdots, p_n)$ and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, and we assume that $p_i > 0$ for every $i \in I$ and $\alpha_1(k) > 0$ for every $k \in K$. For a portfolio $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, the sum $\sum_{i=1}^n x_i\alpha_i$ is its total payoff and $\sum_{i=1}^n x_i p_i$ is its current price.

DEFINITION 1. The law of one price (LOP) holds on the market $M_{p,\alpha}$ if any two portfolios generating the same future payoff have the same price.

The following result is adopted from Chamberlain and Rothschild (1983).

LEMMA 1. *The LOP holds on the market $M_{p,\alpha}$ if and only if there exists $\mu \in M(K)$ such that $\int_K \alpha_i \, d\mu = p_i$ for every $i = 1, \cdots n$.*

3. Measurement of the degree of fulfillment of the LOP

In order to define a measure indicating the degree of fulfillment (or violation) of the LOP accross $M_{p,\alpha}$, consider for every $h = (h_1, \cdots h_n) \in \mathbb{R}^n_+$ the following optimization problem

$$\begin{array}{ll} \text{Maximize} & -\sum_{i=1}^{n} x_i p_i \\ \text{subject to} & \left\{ \begin{array}{ll} \sum_{i=1}^{n} x_i \alpha_i(k) = 0 & \text{for every } k \in K \\ x_i \geq -h_i & i = 1, \cdots n \end{array} \right. \end{array} \qquad [1_h]$$

The problem $[1_h]$ describes the process of identifying the portfolio (constrained by the bounds in a short position $h_i \geq 0$) which minimizes the initial investment needed to purchase a portfolio that generates a zero payoff in every state of nature. Thus, a solution to the problem $[1_h]$ represents the maximum profit obtained by an agent replicating his/her portfolio in such a way that he/she cannot sell more than h_i units in each asset i. When the LOP holds on the market, the optimal value in $[1_h]$ is obviously zero. The following lemma ensures that this maximum arbitrage profit is available.

LEMMA 2. *$[1_h]$ is solvable for every $h \in \mathbb{R}^n_+$.*

Denote by F_h the feasible set of $[1_h]$ and by $\phi(h)$ the optimal value in $[1_h]$, that is,

$$\phi(h) = \max\{-\sum_{i=1}^{n} x_i p_i \mid x \in F_h\}.$$

It is easily verified that

$$\phi(h + h') \geq \phi(h) + \phi(h') \text{ and } \phi(\alpha h) = \alpha \phi(h)$$

for every $h, h' \in \mathbb{R}^n_+$ and $\alpha > 0$, so ϕ is a concave function. In order to prove that ϕ is continuous, we introduce the dual problem for $[1_h]$:

$$\begin{array}{ll} \text{Minimize} & \sum_{i=1}^{n} h_i \lambda_i \\ \text{subject to} & \left\{ \begin{array}{l} \int_K \alpha_i \, d\mu + \lambda_i = p_i \\ \mu \in M(K), \ \lambda_i \geq 0 \quad \text{for every } i = 1, \cdots n \end{array} \right. \end{array} \qquad [2_h]$$

LEMMA 3. *There is strong duality for* $[1_h]$ *(i.e.,*$[1_h]$ *and* $[2_h]$ *are both solvable and there is no duality gap for* $[1_h]$ *and* $[2_h]$*).*

The above lemma is the key to prove the following result.

LEMMA 4. ϕ *is the minimum of some finite number of linear functions. Therefore* ϕ *is a continuous piecewise linear function in* $\mathbb{R}^n_+$.

We can now define a measure as the maximum profit obtained by an investor among all the priced one portfolios h of the bounds in a short position. In these terms the problem is to find $h = (h_1, \cdots h_n) \in \mathbb{R}^n$ so as to

$$\begin{array}{ll} \text{Maximize} & \phi(h) \\ \text{subject to} & \left\{ \begin{array}{ll} \sum_{i=1}^{n} h_i p_i = 1 & \\ h_i \geq 0 & i = 1, \cdots n \end{array} \right. \end{array} \quad [3]$$

Since the feasible set

$$H = \left\{ h \in \mathbb{R}^n \mid \sum_{i=1}^{n} h_i p_i = 1,\ h_i \geq 0,\ i = 1, \cdots n \right\}$$

is compact and ϕ is continuous, the maximum is achieved for some $h^* \in H$. This leads to the following definition.

DEFINITION 2. For a market $M_{p,\alpha}$ satisfying $p_i > 0$, $i = 1, \cdots n$ and $\alpha_1(k) > 0$ for every $k \in K$ the disagreement measure is given by

$$m = \phi(h^*) = \max\{\phi(h) \mid h \in H\}.$$

One can check that, with this definition, m verifies the first requirement to be a measure of non fulfillment of the LOP:

THEOREM 1. *The LOP holds on the market* $M_{p,\alpha}$ *if and only if* $m = 0$.

The requirement of the LOP is thus made testable by estimating m directly. The lower the value of m, the closer the market is to the LOP (i.e., the lower the maximum quotient between the profit and the total price of short-selling restrictions.)

As we will see in Section 6, such a test is also valid for a measurement of market integration: for two or more not perfectly integrated markets (i.e., markets which do not assign the same price to the same future payoff) treated as parts of one combined market, m also indicates the degree of market integration. It is important to point out that m does not depend on the way in which the combined market is divided into smaller groups so that the LOP holds on each one. Only the resultant combined market matters.

Finally, note that it is easy to find out empirically the value of m even in presence of a large set of assets. Just observe that the combined problems $[1_h]$ and [3] are equivalent to the following one

$$\begin{aligned} &\text{Maximize} \quad -\sum_{i=1}^{n} x_i p_i \\ &\text{subject to} \quad \begin{cases} \sum_{i=1}^{n} x_i \alpha_i(k) = 0 & \text{for every } k \in K \\ x_i \geq -h_i & i = 1, \cdots n \\ \sum_{i=1}^{n} h_i p_i = 1 & h_i \geq 0, \quad i = 1, \cdots n \end{cases} \end{aligned} \qquad [4]$$

Problem [4] is a linear program where the decision variable (x, h) takes values in $\mathbb{R}^n \times \mathbb{R}^n_+$. Its solution directly leads to the portfolios x^* and h^*. In practical situations, we can assume that K is a finite set. Then Problem [4] is just a finite-dimensional linear program which can be solved by the classical optimization techniques.

4. A saddle point characterization

Suppose now that the maximum is achieved for $h^* \in H$ and let $x^* \in F_{h^*}$ such that

$$m = \phi(h^*) = -\sum_{i=1}^{n} x_i^* p_i. \qquad [5]$$

From $x_i^* \geq -h_i^*$ one obtains that

$$-\sum_{i=1}^{n} x_i^* p_i \leq \sum_{i=1}^{n} h_i^* p_i = 1,$$

and then

$$0 \leq m \leq 1.$$

The measure m only depends on the current prices p_i and on the prices over the states of nature α. We denote it by $m_{p,\alpha}$ when prices are not fixed. It is easy to check that $m_{p,k\alpha} = m_{p,\alpha}$ for every $k > 0$, so the disagreement measure is current prices relative.

Lemma 3 ensures that $\phi(h)$, the optimal value of $[1_h]$, can be obtained by solving $[2_h]$, that is,

$$\phi(h) = \min_{l\in} \sum_{i=1}^{n} h_i \lambda_i ,$$

where

$$\Lambda = \left\{ \lambda \in \mathbb{R}^n_+ \mid \int_K \alpha_i \, d\mu + \lambda_i = p_i, \ i = 1, \cdots n, \ \mu \in M(K) \right\}.$$

Hence, the problem of finding m can be expressed by a max-min problem

$$m = \max_{h\in H} \min_{\lambda\in\Lambda} U(\lambda, h) ,$$

where U is defined by $U(\lambda, h) = \sum_{i=1}^{n} h_i \lambda_i$.

THEOREM 2. *For the convex-concave function $U(\lambda, h)$ defined above there exists a saddle point (λ^*, h^*) and*

$$m = \max_{h\in H} \min_{\lambda\in\Lambda} U(\lambda, h) = U(\lambda^*, h^*) = \min_{\lambda\in\Lambda} \max_{h\in H} U(\lambda, h).$$

In particular, the LOP holds on the market $M_{p,\alpha}$ if and only if the function $U(\lambda, h)$ possesses a saddle point at $(0, h)$ for every $h \in H$.

In game theoretic terminology the equality in the above Theorem expresses a two-person zero-sum game of the investor against the "market". Since $\lambda_i = p_i - \int_K \alpha_i \, d\mu$ could be interpreted as the error commited by the "market" in the price of each asset for the state prices μ, the sum $\sum_{i=1}^{n} h_i \lambda_i$ would be the payment from the "market" to the investor due to h and λ. Thus, the investor chooses a priced one portfolio of short-selling bounds in such a way that it maximizes the minimal payment desired by the "market" and solves $\max_{h\in H} \min_{\lambda\in\Lambda} U(\lambda, h)$. The problem, $\min_{\lambda\in\Lambda} \max_{h\in H} U(\lambda, h)$ describes the process by which the "market" counteracts the goal of the investor by choosing the feasible λ (i.e., the feasible implicit state prices μ) which minimizes the maximal payment desired by the investor.

5. Other representations of the disagreement measure

In the previous sections we defined the disagreement measure to be the maximum achieved in the optimization problem [3], which represents the maximum profit with short-selling restrictions, stocks for example, with total current price one. With the aid of some lemmas, we can also express the disagreement measure in two alternative ways.

First, the measure represents the maximum profit obtained from a portfolio relative to the price of the sold assets. Second, we establish a relation between the disagreement measure and the maximum profit obtained from a portfolio relative to the price of all exchanged (sold and purchased) assets. This relation could be useful to study arbitrage opportunities in economies with frictions.

LEMMA 5. *Given a market $M_{p,\alpha}$, let $h^* \in H$ and $x^* \in F_{h^*}$ verifying* [5], *and suppose $\phi(h^*) > 0$. Then*
i) $x_j^* = -h_j^*$ or $h_j^* = 0$ (or both) for every $j = 1, \cdots n$.
ii) In particular, if $x_j^* > -h_j^*$ then $x_j^* > 0$.

Lemma 5 ii) says basically that the portfolio where the maximum profit is achieved either sells all the stock or purchases in each asset.

For a portfolio $x \in \mathbb{R}^n$, denote by S_x the set of indices of the sold assets and by L_x the set of indices of the purchased assets, that is, $S_x = \{i = 1, \cdots n \mid x_i < 0\}$ and $L_x = \{i = 1, \cdots n \mid x_i \geq 0\}$.

THEOREM 3. *Assume the existence of a portfolio $x \in \mathbb{R}^n$ such that*

$$\sum_{i=1}^{n} x_i p_i < 0 \text{ and } \sum_{i=1}^{n} x_i \alpha_i(k) = 0$$

for every $k \in K$. Then
i) There exists $i \in \{1, 2, \cdots n\}$ such that $x_i < 0$.
ii) $\frac{\sum_{i=1}^{n} x_i p_i}{\sum_{i \in S_x} x_i p_i} \leq \phi(h^*)$. In particular $\phi(h^*) > 0$.

iii) If $x = x^*$ then $\frac{\sum_{i=1}^{n} x_i p_i}{\sum_{i \in S_x} x_i p_i} = \phi(h^*)$.

Theorem 3 proves that when LOP fails x^* solves the following optimization problem

$$\begin{aligned} &\text{Maximize} \quad \frac{-\sum_{i=1}^{n} x_i p_i}{-\sum_{i \in S_x} x_i p_i} \\ &\text{subject to} \quad \begin{cases} \sum_{i=1}^{n} x_i \alpha_i(k) = 0 & \text{for every } k \in K \\ \sum_{i=1}^{n} x_i p_i < 0 \end{cases} \end{aligned} \qquad [6]$$

and $\phi(h^*)$ is then the objective optimal value.

This result is mathematically interesting by itself, because it solves an optimization problem with strict inequality constraints and a non differentiable objective. Furthermore, it gives another representation of the disagreement measure in a market without short-selling restrictions as the maximum profit obtained from a portfolio with total sold assets price one (observe that

$$f(x) = \frac{-\sum_{i=1}^{n} x_i p_i}{-\sum_{i \in S_x} x_i p_i}$$

is a homogeneous function of degree zero and for every feasible x, kx is also feasible for every $k > 0$).

Theorem 3 yields some important properties. The complete relaxation of the constraints imposed to the short positions leads to the same measure m. Hence, an agent can replicate his/her portfolio in an arbitrary way and m still tests the maximum relative profit over the total price of the sold assets. Choosing a portfolio x where this maximum profit is achieved, we can deduce from Lemma 5 how must one choose h^* and take into account short-selling restrictions. That is,

$$h_i^* = 0 \text{ if } x_i^* \geq 0 \text{ and } h_i^* = \frac{x_i^*}{\sum_{i \in S_x^*} x_i^* p_i} \text{ if } x_i^* < 0.$$

Another and very natural measure arises in order to capture the degree of non fulfillment of the Law of One Price. It is important to know how do the taxes (or transaction costs) affect to the fulfillment of the Law of One Price. A possible way is to compute the minimum rate of taxes l that is necessary to get rid of all available arbitrage opportunities. Formally, it is the minimum l' such that

$$(1+l')\left(\sum_{i \in L_x} x_i p_i\right) \geqslant (1-l')\left(-\sum_{i \in S_x} x_i p_i\right)$$

for every $x \neq 0$ such that $\sum_{i=1}^{n} \alpha_i(k) = 0$, $k \in K$. Latter inequality is equivalent to

$$l'\left(\sum_{i=1}^{n} |x_i| p_i\right) \geqslant -\sum_{i=1}^{n} x_i p_i.$$

This leads us to consider for every portfolio $x \neq 0$ the function

$$g(x) = \frac{-\sum_{i=1}^{n} x_i p_i}{-\sum_{i \in S_x} x_i p_i + \sum_{i \in L_x} x_i p_i} = \frac{-\sum_{i=1}^{n} x_i p_i}{\sum_{i=1}^{n} |x_i| p_i}. \qquad [7]$$

The function $g(x)$ is the quotient between the profit generated by the zero payoff x and the price of all interchanged assets. Moreover, l is just the solution of

$$\text{Maximize} \quad g(x)$$
$$\text{subject to} \quad \begin{cases} \sum_{i=1}^{n} x_i \alpha_i(k) = 0 & \text{for every } k \in K \\ x \neq 0 \end{cases} \qquad [8]$$

Manipulating $g(x)$, it is easy to prove that

$$g(x) = \frac{f(x)}{2 - f(x)} \text{ for every } x \in \mathbb{R}^n \text{ such that } \sum_{i \in S_x} x_i p_i \neq 0 \,.$$

These observations and the fact that $\frac{t}{2-t}$ is a increasing continuous function in $[0\,,\,1]$ $(0 \leq m \leq 1)$ lead to the following theorem.

THEOREM 4. *Let $m > 0$ and consider $h^* \in H$ and x^* such that*

$$m = \frac{\sum_{i=1}^{n} x_i^* p_i}{\sum_{i \in S_{x^*}} x_i^* p_i} = \phi(h^*).$$

Then, x^* solves problem [8].

We have then proved that problem [8] is solvable when the LOP does not hold on the market. Obviously, if the LOP holds on the market and [8] is feasible then $g(x) = 0$ for every feasible x in [8] and, [8] is also solvable (the optimal value is then zero.) Observe that l, the optimal solution in [8], satisfies

$$l = g(x^*) = \frac{f(x^*)}{2 - f(x^*)} = \frac{m}{2 - m}, \qquad [9]$$

whenever the LOP does not hold on the market. The relation $l = \frac{m}{2-m}$ remains also true when the LOP holds on the market. Thus, l is also a measure of the degree of discrepancy of prices. It is also a relative measure which takes values in $[0, 1]$ and such that $l \leq m$, since $0 \leq m \leq 1$.

In this case the measure l computes the maximum profit relative to the total price of the exchanged assets. As already said, it could be tested if the market frictions affect to the fulfillment of the LOP. In fact, assume that the transaction costs are determined by the price V of the exchanged assets. The maximum profit an investor can obtain for V is lV. Then, we can discount the transaction costs from lV and verify if the LOP holds.

Jouini and Kallal (1995) (see also Jouini and Kallal (1992/1)) follow a different approach to introduce market frictions (taxes, transaction costs, bid-ask spread $\cdots$). They assume the existence ot two prices per security $v_i \leq c_i$, $i = 1, 2, \cdots, n$, in such a way that the investors pay c_i in purchasing the i^{th} asset while they sell it with the price v_i. Thus, the current price of a given portfolio x is just $q_1x_1 + q_2x_2 + \cdots + q_nx_n$ where $q_i = c_i$ if $x_i \geq 0$ and $q_i = v_i$ if $x_i < 0$. Hence, the prices are not given by linear functions but by sublinear and convex ones. Under these assumptions, and bearing in mind that the outcome states are given by the compact set K, the absence of arbitrage is characterized by the existence of a positive measure $\mu \in M(K)$ such that $v_i \leq \int_K \alpha_i \, d\mu \leq c_i$ for every $i = 1, 2, \cdots, n$. They also extend latter result to a model where there are two payoffs per security. The theory we have developed may be very easily extended in order to introduce the measures m and l after incorporating Jouini and Kallal ideas. In such a case $[1_h]$ becomes a convex program instead of a linear one while its dual is a minor modification of program $[2_h]$. In any case our main results still hold since the duality theory is also generalized from linear to convex programming (see for instance Balbás and Guerra (1996)).

The results in this section show that the non-differentiable programs [6] and [8] may be solved applying the saddle point condition in Theorem 2 or solving the (linear) Program [4]. Besides, the measures m and l do not depend on the short-selling restrictions imposed in order to avoid mathematical difficulties.

6. Applications to financial market integration

Chen and Knez (1995) develop a measurement theory of market integration for two markets whenever there exist discrepancies in pricing common asset payoffs or, equivalently, when the LOP is violated across them. They assume the LOP holds separately on each market and use a model slightly different to ours: they consider the linear space of square-integrable random variables L^2 over a probality space (Ω, F, P_R) instead of $C(K)$. Readapting it to the mathematical setting of this paper, they define what they call the weak-integration measure $g(M_1, M_2)$ as the minimum distance between the sets of state prices

$$D_1 = \left\{ \mu \in M(K) \mid \int_K \alpha_i \, d\mu = p_i \qquad i = 1, \cdots q \right\}$$

and

$$D_2 = \left\{ \mu \in M(K) \mid \int_K \alpha_i \, d\mu = p_i \qquad i = q+1, \cdots n \right\},$$

where $M_1 = M_{p^1,\alpha^1}$, $M_2 = M_{p^2,\alpha^2}$, $p^1 = (p_1, \cdots p_q)$, $\alpha^1 = (\alpha_1, \cdots \alpha_q)$ $p^2 = (p_{q+1}, \cdots p_n)$ and $\alpha^2 = (\alpha_{q+1}, \cdots \alpha_n)$. Thus,

$$g(M_1, M_2) = \inf\{\|\mu_1 - \mu_2\| \mid \mu_1 \in D_1, \mu_2 \in D_2\}.$$

Chen and Knez (1995) also prove that the weak integration measure equals the maximum difference between the respective prices assigned by both markets to any unit-norm payoff.

The measure m is an alternative to the weak-integration measure. Treating both markets M_1 and M_2 as parts of the combined market $M_{p,\alpha}$, where $p = (p_1, \cdots p_q, p_{q+1} \cdots p_n)$ and $\alpha = (\alpha_1, \cdots \alpha_q, \alpha_{q+1} \cdots \alpha_n)$, we compute m on this global market. Note that m can be computed even if both markets do not separately verify the LOP. In such a case m gives no indication as to whether this maximum relative profit comes from each market violations, or cross-market violations. We estimate m_1 on M_1, m_2 on M_2 and m on the combined market. Then,

$$m - \max(m_1, m_2)$$

reveals which part of the maximum relative profit is due to the cross-markets violations. Hence, m reflects a degree of integration of two or more markets verifying the LOP or not.

Theorem 5 reveals that m is continuous with respect to $g(M_1, M_2)$, and gives an upper bound for m which depends only on the returns of the different assets available on each market.

THEOREM 5. *The following inequalities hold:*

i) $l \leq m \leq \sum_{j=1}^{q} \frac{\|\alpha_j\|}{p_j} g(M_1, M_2)$

ii) $l \leq m \leq \sum_{j=q+1}^{n} \frac{\|\alpha_j\|}{p_j} g(M_1, M_2)$

iii) $l \leq m \leq \frac{1}{2} \sum_{j=1}^{n} \frac{\|\alpha_j\|}{p_j} g(M_1, M_2)$

Remarks. 1) Readapting our model to Chen and Knez's model, the same inequalities would be obtained writing $\|\alpha_j\|_2$ instead of $\|\alpha_j\|$ for every $j = 1, \cdots n$ where $\|\alpha_j\|_2^2 = \int \alpha_j^2 \, dP_R$.

2) Theorem 5 iii) remains true for any division of the combined market into two markets in such a way that the assets in each group satisfy the LOP.

The continuity of g with respect to m can not be proved. The following examples show this and also illustrate some differences between both integration measures.

Example 1. Consider the case where there are two possible states of nature, s_1 and s_2. Suppose one asset A_1 paying \$ 1 in s_1 and \$ 0 in s_2 with a current price of \$ 1, another asset A_2 paying \$ 0 in s_1 and \$ 1 in s_2, with a current price of \$ 1, and a third one A_3 paying \$ 1 in s_1 and \$ α in s_2 with a current price of \$ p ($p > 0$).

Direct calculations solving [3] or [4] show that for $M = \{A_1, A_2, A_3\}$ the disagreement measure is given by

$$m = \frac{|1 + \alpha - p|}{\max(1 + \alpha, p, p - \alpha)}.$$

We compute g for some of the different ways of dividing the market M into two markets so that the LOP holds separately on each market.

i) For $M_1 = \{A_1, A_2\}$ and $M_2 = \{A_3\}$ we get $g(M_1, M_2) = \frac{|1+\alpha-p|}{\sqrt{1+\alpha^2}}$.

ii) For $S_1 = \{A_1\}$ and $S_2 = \{A_2, A_3\}$ we get $g(S_1, S_2) = |1 + \alpha - p|$.

iii) For $N_1 = \{A_2\}$ and $N_2 = \{A_1, A_3\}$ (note that we need $\alpha \neq 0$, or $p = 1$ if $\alpha = 0$) we get $g(N_1, N_2) = \frac{|1+\alpha-p|}{|\alpha|}$ if $\alpha \neq 0$ and $g(N_1, N_2) = 0$ otherwise.

iv) For $H_1 = \{A_1, A_3\}$ and $H_2 = \{A_2, A_3\}$ and the same assumptions for α and p as in iii) we get $g(H_1, H_2) = \frac{|1+\alpha-p|}{|\alpha|}(\sqrt{1 + \alpha^2})$ if $\alpha \neq 0$ and $g(H_1, H_2) = 0$ otherwise.

This shows that the weak-integration measure g depends on the way in which the market M is divided. If we compare the results obtained in ii) and iii) for a fixed price p, we observe that

$$\lim_{\alpha\to\infty} g(S_1, S_2) = \infty\,, \text{ although } \lim_{\alpha\to\infty} g(N_1, N_2) = 1.$$

Example 2. Consider the case where there are two possible states of nature, s_1 and s_2. Suppose one asset A_1 paying \$1 in both states and with current price of \$1, another asset A_2 paying $\$1+\alpha$ in s_1 and $\$1-\alpha$ in s_2 with a current price of \$1, and a third one A_3 paying $\$1+2\alpha$ in s_1 and $\$1-\alpha$ in s_2 with a current price of \$1.

Dividing the market M into the markets $J_1 = \{A_1, A_2\}$ and $J_2 = \{A_1, A_3\}$ we get

$$g(J_1, J_2) = \frac{\sqrt{2}}{6} \text{ if } \alpha \neq 0$$
$$\text{and } g(J_1, J_2) = 0 \text{ if } \alpha = 0.$$

First, note that both markets verify the LOP, and also that no arbitrage opportunity exists in each market, in the sense that there are no positive payoffs with negative or zero prices, since for each market there exists a positive state price. Second, the weak-integration measure is not continuous in α. Intuitively, for both markets J_1 and J_2, the closer to 0 the value of α, the more closely integrated the two markets are Direct calculations show that for $M = \{A_1, A_2, A_3\}$ and $\alpha \in [0\,,\,1)$ the value of m is $\frac{\alpha}{3}$, which is continuous in α.

We conclude from the two examples above that m and g reflect different things and therefore both measures must be considered in analyzing the integration between markets. Furthermore, m gives an information in monetary terms, an useful fact for the investors.

7. Conclusions

This paper presents two measures (m and l) of the degree of fulfillment of the Law of One Price (LOP) in a financial market. These measures are related by [9] and vanish if and only the LOP holds on the market. When the LOP does not hold, the measures are strictly positive, and they increase if the level of violation of the LOP increases, that is, when there is an increase in the profit that an investor can obtain relative to the total amount of money he/she has to exchange replicating his/her portfolio and implementing the arbitrage. The maximum value of these measures is one, and it is attained in extreme situations in which the agents can replicate their portfolios in such a way that they obtain a profit equal to the price of the sold assets. This is a limiting case which will never appear in practical situations.

The measures have a certain a number of interesting properties. Among them, they are continuous with respect to the prices and the payoffs, they provide the relative bid-ask spread when there is a unique asset, and they increase (or at least they do not decrease) when new assets are introduced. Moreover, the measures do not depend on the short-selling restrictions in the model. We may assume either that the agents cannot replicate by selling the assets they do not have, or the opposite, that is, there is no limit on the short positions the agents can hold. In both cases the same values for both measures are obtained. These results are mathematically interesting by themselves since they provide a procedure to solve a non-differentiable optimization problem.

The LOP is usually characterized by the existence of state prices, but this is a very specific criterium which only determines the accomplishment or not of the LOP. A measure able to reflect the degree of fulfillment, that is, the maximum relative profits available by the agents, makes the model more flexible, and therefore, more realistic. If the measure is strictly positive and small enough, we are most likely to have an efficient market, although either the trading frictions and transaction costs, or the measurement errors may lead to this positive value.

Our measures also allow us to analyze markets with frictions. In fact, since they quantify the degree of fulfillment of the LOP in (relative) monetary terms, once we have computed them we can discount the transaction costs and, therefore, we may know if the agents can implement the arbitrage opportunities.

Many empirical papers analyze some well-known specific arbitrage strategies. Our measures have the important advantage that in computing them, all the assets in the market are selected and, therefore, all the arbitrage opportunities are considered.

In considering several financial markets, one can work in a global market collecting all the available assets from the (sub)markets. Then m and l (computed on the combined market) may be taken to measure the integration of all them. These new market integration measures are continuous with respect to the Chen and Knez measure g, so estimating and controlling their measure, our measures are also controlled. The opposite is in general false, and there are situations in which m takes small values while g remains large. This is an important fact, once one realizes that m is qualitatively different from g, since it measures in monetary terms (relative profits available by the agents). Furthermore, m seems to be useful to estimate the degree of multimarket integration in monetary terms.

The theory developed above analyzes the degree of fulfillment or violation of the LOP in a financial market or across different ones, but it may be easily extended in two directions: First, in order to reflect the level of arbitrage opportunities in the strong form (or cross-market arbitrage) and second in order to consider the imperfect financial market case and incorporate the bid-ask spread or short-selling costs.

Appendix

PROOF OF LEMMA 2. The proof of the lemma relies on the fact that if the maximum is achieved for some feasible $x^* = (x_1^*, \cdots, x_n^*)$ then

$$x_j^* \leq \frac{\sum_{i \neq j} p_i h_i}{p_j} = \beta_j \text{ for every } j, \qquad [A1]$$

and the original problem is equivalent to

$$\begin{array}{ll} \text{Maximize} & -\sum_{i=1}^{n} x_i p_i \\ \text{subject to} & \left\{ \begin{array}{ll} \sum_{i=1}^{n} x_i \alpha_i(k) = 0 & \text{for every } k \in K \\ \beta_i \geq x_i \geq -h_i & i = 1, \cdots n \end{array} \right. \end{array} \qquad [A2_h]$$

The feasible set of $[A2_h]$ is non void and compact. Hence, there exists an optimal solution to $[A2_h]$ and, consequently, to $[1_h]$.
It only remains to prove that $[A1]$ holds. In fact, since $x^* \in F_h$, we have $x_j^* \geq -h_j$ for every $j = 1, \cdots n$. Besides, the optimal value in $[1_h]$ is greater than 0, since the zero vector is in F_h. Combining these two facts, $[A1]$ is easily derived. □

PROOF OF LEMMA 3. In the inequality-constrained program $[1_h]$, $x \in \mathbb{R}^n$ and the associated positive cone P is $\mathbb{R}^n$, while the inequality constraints take values in $C(K) \times \mathbb{R}^n$, where the associated positive cone Q is $\{0\} \times \mathbb{R}^n$. P is a cone with compact sole (since $B = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ is a compact set in P such that 0 is not in B and B spans P.) Besides, if $x \in \mathbb{R}^n$ is such that $(-\sum_{i=1}^n x_i \alpha_i, x) \in Q$ and $\sum_{i=1}^n x_i p_i = 0$ then $x = 0$.

These two facts allow us to ensure that

$$D' = \left\{ \left(-\sum_{i=1}^{n} x_i \alpha_i,\ x - y,\ \sum_{i=1}^{n} x_i p_i \right) \mid x \in \mathbb{R}^n,\ y \in \mathbb{R}^n_+ \right\}$$

is a closed set (see Theorem 3.19 of Anderson and Nash(1987)). Now, from Theorems 3.10 and 3.22 (Anderson and Nash(1987)), Lemma 3 is deduced. □

PROOF OF LEMMA 4. Lemma 3 ensures that $\phi(h)$ is the optimal value of $[2_h]$. Denoting by T the linear map $T(\mu) = (\int_K \alpha_i \, d\mu)_{i=1}^n$ from $M(K)$ to $\mathbb{R}^n$, it follows that $T(M(K))$ is a vector space in $\mathbb{R}^n$. So, the feasible set in $[2_h]$ is the intersection between $p - T(M(K))$ and $\mathbb{R}^n_+$. Therefore, there exists only a finite number of extreme points $\lambda^1, \lambda^2 \cdots \lambda^r$ for this feasible set which do not depend on h, since the feasible set does not depend on h. Then

$$\phi(h) = \min \left(\sum_{i=1}^{n} h_i \lambda_i^1, \sum_{i=1}^{n} h_i \lambda_i^2, \cdots, \sum_{i=1}^{n} h_i \lambda_i^r \right)$$

and the proof is concluded. □

PROOF OF THEOREM 1. Just observe that the LOP holds on the market if and only if $\sum_{i=1}^{n} x_i p_i = 0$ whenever $\sum_{i=1}^{n} x_i \alpha_i(k) = 0$ for every $k \in K$ or, equivalently, if $\phi(h) = 0$ for every $h \in \mathbb{R}^n_+$. But taking into account that ϕ is homogeneous of degree one, the latter is equivalent to $m = 0$. □

PROOF OF THEOREM 2. For every $\lambda \in \Lambda$, denote

$$f(\lambda) = \max_{h \in H} \sum_{i=1}^{n} h_i \lambda_i.$$

Since H is a compact set, such a maximum exists. Moreover, the maximum is achieved for some extreme point $A_i(0, \cdots, \frac{1}{p_i}, \cdots 0)$, and then $f(\lambda) = \max(\frac{\lambda_1}{p_1}, \frac{\lambda_2}{p_2} \cdots \frac{\lambda_n}{p_n})$. Hence f is a continuous function. Since $f(\lambda) \geq \phi(h)$ for every $\lambda \in \Lambda$ and $h \in H$, the function ϕ is a bounded. Thus, there exists $\beta = \inf\{f(\lambda) \mid \lambda \in \Lambda\}$.

Note that H and Λ are convex subsets, H is a compact set, $U(\lambda, .)$ is quasiconcave and upper-semicontinuous for every $\lambda \in \Lambda$, and $U(., h)$ is quasiconvex and below-semicontinuous for every $h \in H$. Hence Sion's theorem (Moulin(1979)) yields

$$\beta = \inf\{f(\lambda) \mid \lambda \in \Lambda\} = \max\{\phi(h) \mid h \in H\}.$$

It only remains to prove that such an infimum is achieved for some $\lambda^* \in \Lambda$. To do this, take a sequence $(\lambda^k)_{k=1}^{\infty}$ in such that $f(\lambda^k) < \beta + \frac{1}{k}$ for every k. Then $f(\lambda^k) < \beta + 1$ and since $f(\lambda^k) = \max(\frac{\lambda_1^k}{p_1}, \cdots, \frac{\lambda_n^k}{p_n})$ it follows that $\lambda_i^k < p_i(\beta + 1)$ for every $k \in \mathbb{N}$ and $i \in \{1, 2 \cdots n\}$. Thus, $(\lambda^k)_{k=1}^{\infty}$ is a bounded sequence in $\mathbb{R}_+^n$ and, consequently, there exists a subsequence converging to λ^*, which we denote also by $(\lambda^k)_{k=1}^{\infty}$. Since Λ is a closed set, we have $\lambda^* \in \Lambda$. From the continuity of f and the choice of the sequence $(\lambda^k)_{k=1}^{\infty}$, the inequality $f(\lambda^*) \leq \beta$ is derived and, then, $f(\lambda^*) = \beta$. Taking now $h^* \in H$ such that $\phi(h^*) = f(\lambda^*)$, it follows that (λ^*, h^*) is a saddle point for U. □

PROOF OF LEMMA 5. ii) is easily deduced from i). Let us prove i).

In order to simplify the notation and without loss of generality, we will prove i) for $j = n$. Proceeding by contradiction, suppose $x_n^* > -h_n^*$, and $h_n^* > 0$.

First, assume $x_n^* \geq 0$, and let $h_0 = (h_1^*, \cdots, h_{n-1}^*, 0)$ and $\gamma = \sum_{i=1}^{n-1} p_i h_i^*$. It is easily proved that $0 < \gamma < 1$. Besides $\phi(h_0) \leq \phi(h^*)$, since the feasible sets F_{h_0} and F_{h^*} are such that $F_{h_0} \subseteq F_{h^*}$. From $x^* \in F_{h_0}$ we get $\phi(h^*) = \phi(h_0)$. Finally, we have

$$\phi(h^*) = \phi(h_0) = \gamma\phi(\frac{1}{\gamma}h_0) < \phi(\frac{1}{\gamma}h_0) \leq \phi(h^*), \qquad [A3]$$

where the last inequality follows from the fact that $\frac{1}{\gamma}h_0 \in H$. Since $[A3]$ leads to a contradiction, the proof is concluded whenever $x_n^* \geq 0$.

Assume now $-h_n^* < x_n^* < 0$ and let $h^0 = (h_1^*, \cdots, h_{n-1}^*, |x_n^*|)$ and $\delta = \sum_{i=1}^{n-1} p_i h_i^* + p_n |x_n^*|$. As above, we get $0 < \delta < 1$ and

$$\phi(h^*) = \phi(h^0) = \delta\phi(\frac{1}{\delta}h^0) < \phi(\frac{1}{\delta}h^0) \leq \phi(h^*),$$

which concludes the proof of Lemma 5. □

PROOF OF THEOREM 3. i) follows from the fact that $\sum_{i=1}^{n} x_i p_i \geq 0$ whenever $x_i \geq 0$ for every $i \in \{1, 2 \cdots n\}$.

ii) Let $h' = (h_1', h_2' \cdots h_n')$, where $h_i' = \max(-x_i, 0)$ for every $i = 1, \cdots n$, and $\varepsilon = \sum_{i=1}^{n} p_i h_i' = -\sum_{i \in S_x} x_i p_i > 0$.

Since $\frac{1}{\varepsilon}h' \in H$ we get $\phi(\frac{1}{\varepsilon}h') = \frac{1}{\varepsilon}\phi(h') \leq \phi(h^*)$.

Besides, from $x \in F_{h'}$ we get $-\sum_{i=1}^{n} p_i x_i \leq \phi(h')$.

Thus, combining both inequalities, ii) is proved.

iii) Assume $x = x^*$. Then $\phi(h^*) = -\sum_{i=1}^{n} x_i^* p_i$. From Lemma 5 we get $x_i^* = -h_i$ whenever $i \in S_{x^*}$ and $h_i^* = 0$ otherwise. Thus,

$$-\sum_{i \in S_{x^*}} x_i^* p_i = \sum_{i \in S_{x^*}} h_i^* p_i = \sum_{i=1}^{n} h_i^* p_i = 1$$

and the proof of iii) is concluded. □

PROOF OF THEOREM 5. Proofs of i) and ii) are similar, and iii) is an easy consequence of i) and ii). So, we will just prove i).

Let $h^* \in H$, $x^* \in F_{h^*}$ such that

$$m = \phi(h^*) = -\sum_{i=1}^{n} x_i^* p_i .$$

From Lemma 2, x^* is also a feasible optimal solution of

$$\begin{array}{ll} \text{Maximize} & -\sum_{i=1}^{n} x_i p_i \\ \text{subject to} & \begin{cases} \sum_{i=1}^{n} x_i \alpha_i(k) = 0 & \text{for every } k \in K \\ \beta_i \geq x_i \geq -h_i^* & i = 1, \cdots n \end{cases} \end{array} \qquad [\mathrm{A2}_{h^*}]$$

where

$$\beta_i = \frac{\sum_{j \neq i} p_j h_j^*}{p_i} = \frac{1 - p_i h_i^*}{p_i} = \frac{1}{p_i} - h_i^* .$$

The dual problem of $[A2_{h^*}]$ is given by

$$\begin{array}{ll} \text{Minimize} & \sum_{i=1}^{n} (\frac{1}{p_i} - h_i^*)\lambda_i^- + \sum_{i=1}^{n} h_i^* \lambda_i^+ \\ \text{subject to} & \begin{cases} \int_K \alpha_i \, d\mu - \lambda_i^- \quad + \lambda_i^+ = p_i \\ \lambda_i^+ \geq 0, \lambda_i^- \geq 0 \quad i = 1, \cdots n \text{ and } \mu \in M(K) \end{cases} \end{array} \qquad [\mathrm{A4}_{h^*}]$$

Denoting

$$I^+ = \{i \in \{1, 2, \cdots n\} \mid p_i - \int_K \alpha_i \, d\mu \geq 0\}$$

and

$$I^- = \{i \in \{1, 2 \cdots n\} \mid p_i - \int_K \alpha_i \, d\mu < 0\},$$

problem $[A4_{h^*}]$ can be reformulated as

$$\begin{array}{ll} \text{Minimize} & \sum_{i \in I^-} \left(\frac{1}{p_i} - h_i^*\right) \left(\int_K \alpha_i \, d\mu - p_i\right) + \sum_{i \in I^+} h_i^* \left(p_i - \int_K \alpha_i \, d\mu\right) \\ \text{subject to} & \mu \in M(K) \end{array}$$

and then,

$$m \leq \sum_{i \in I^-} \left(\frac{1}{p_i} - h_i^*\right) \left(\int_K \alpha_i \, d\mu - p_i\right) + \sum_{i \in I^+} h_i^* \left(p_i - \int_K \alpha_i \, d\mu\right). \tag{A5}$$

Now, set $g = g(M_1, M_2)$ and let $\varepsilon > 0$. Choose $\mu_1 \in D_1$ and $\mu_2 \in D_2$ such that

$$g \geq \|\mu_1 - \mu_2\| - \varepsilon = \sup \left\{ \left| \int_K \alpha \, d(\mu_1 - \mu_2) \right| \mid \alpha \in C(K), \, \|\alpha\| \leq 1 \right\} - \varepsilon.$$

Then,

$$g \geq \frac{1}{\|\alpha_i\|} \left| \int_K \alpha_i \, d\mu_1 - \int_k \alpha_i \, d\mu_2 \right| - \varepsilon$$

for every $i = 1, \cdots n$. Since $\int_K \alpha_j \, d\mu_1 = p_j$ for $j = 1, \cdots q$, we then have

$$\|\alpha_j\| g \geq \left| \int_K \alpha_j \, d\mu_2 - p_j \right| - \|\alpha_j\| \varepsilon \tag{A6}$$

for every $j = 1, \cdots q$. Denoting

$$J^+ = \{j \in \{1, 2, \cdots q\} \mid p_j - \int_K \alpha_j \, d\mu_2 \geq 0\}$$

and

$$J^- = \{j \in \{1, 2 \cdots q\} \mid p_j - \int_K \alpha_j \, d\mu_2 < 0\},$$

we multiply [15] by h_j^* for every $j \in J^+$, and by $\frac{1}{p_j} - h_j^*$ for every $j \in J^-$. Adding up all the obtained inequalities, we get

$$Ag \geq \sum_{j \in J^+} h_j^* (p_j - \int_K \alpha_j \, d\mu_2) + \sum_{j \in J^-} (\frac{1}{p_j} - h_j^*)(\int_K \alpha_j \, d\mu_2 - p_j) - A\varepsilon, \tag{A7}$$

where $A = \sum_{j \in J+} \|\alpha_j\| h_j^* + \sum_{j \in J-} \|\alpha_j\| (\frac{1}{p_j} - h_j^*)$. Since $h_i^* \geq 0$ and $\frac{1}{p_i} - h_i \geq 0$ for every $i = 1, 2 \cdots q$, it follows that

$$A \leq \sum_{i=1}^{q} \|\alpha_i\| h_i^* + \sum_{i=1}^{q} \|\alpha_i\| (\frac{1}{p_i} - h_i^*) = \sum_{i=1}^{q} \frac{\|\alpha_i\|}{p_i}. \qquad [A8]$$

Besides, from the fact that $\int_K \alpha_i \, d\mu_2 = p_i$ for every $i = q + 1, \cdots n$, we obtain that

$$\sum_{j \in J+} h_j^* \left(p_j - \int_K \alpha_j \, d\mu_2 \right) + \sum_{j \in J^-} (\frac{1}{p_j} - h_j^*) \left(\int_K \alpha_j \, d\mu_2 - p_j \right) \qquad [A9]$$

$$= \sum_{i \in I+} h_i^* \left(p_i - \int_K \alpha_i \, d\mu_2 \right) + \sum_{i \in I^-} (\frac{1}{p_i} - h_i^*) \left(\int_K \alpha_i \, d\mu_2 - p_i \right).$$

From [A5], [A7], [A8], and [A9] it follows that

$$\sum_{i=1}^{q} \frac{\|\alpha_i\|}{p_i} g \geq m - \varepsilon \sum_{i=1}^{q} \frac{\|\alpha_i\|}{p_i}$$

and then, Theorem 5 i) is proved. □

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Resumen

En el presente trabajo se introducen dos nuevas medidas del nivel de cumplimiento de la Ley de Precio Único. Puesto que las medidas se definen a través de programas de optimización cuya función objetivo es no diferenciable, se refomulan los problemas con objeto de caracterizarlas mediante condiciones de punto de silla. Muchos trabajos empíricos contrastan estrategias de arbitraje concretas y bien conocidas. Mediante la aplicación de nuestras medidas el enfoque es esencialmente diferente, ya que el mercado y sus oportunidades de arbitraje quedan analizados de forma global, y se va mucho más allá de la contrastación de técnicas y estrategias concretas. Este enfoque global es también muy útil para estudiar niveles de integración de diferentes mercados financieros, o mercados con fricciones. Se relacionan las medidas con otras ya existentes en la literatura, y se observa como nuestro enfoque permite traducir en términos monetarios el nivel de incumplimiento, si lo hay, de la Ley de Precio Único. La teoría desarrollada puede extenderse fácilmente para medir en términos monetarios los niveles de las oportunidades de arbitrage existentes incluso en mercados que cumplen la Ley de Precio Único.