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THE ECONOMETRICS OF THE STOCK MARKET II: ASSET PRICING

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This paper reviews three empirically orientated asset pricing models: the consumption and traditional CAPM, and the APT. They all imply that asset prices are the expected value of their payoffs times a common stochastic discount factor. Thus, an asset risk premium is proportional to its covariance with economy wide risks and diversifiable risk is not priced.
Models are usually tested by checking if the implied discounted excess returns are zero on average for several assets simultaneously. The empirical success is higher for those that explain returns in terms of other assets instead of «macroeconomic» variables. Nevertheless, there are several unexplained anomalies.

1. Introduction

Asset pricing theories are concerned with determining the expected returns of assets whose payoffs are risky. These include not only financial securities such as stocks and bonds, but also real assets such as venture capital. Explicitly, these models analyze the relationship between risk and expected return, and address the crucial question of how to value risk. In this paper, we shall review three of the most popular empirically orientated asset pricing models currently available, the traditional Capital Asset Pricing Model (CAPM), the Arbitrage Pricing Theory (APT) and the Consumption CAPM. Importantly, we shall consider dynamic versions of these theories in which changing information implies a changing risk perception by agents.

Our analysis will be done in discrete time in order to avoid the use of the tools of stochastic calculus necessary for intertemporal continuous time asset pricing models. As a result, we have to ommit a significant fraction of the theoretical literature (see Merton (1990) and the references therein),

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and an increasing number of applied papers. We shall not consider either asymmetric information models in which different agents have access to different information, or models in which agents hold different views about the world.

Let R_{jt} be the gross return of asset j during period t, that is, the total payoff (price and «dividend») that we would receive at the end of period t if we were to invest one unit (of some numeraire good) in asset j today (end of period $t-1$). Since asset j is risky, its return, R_{jt}, a bundle of contingent claims, is random and not known at $t-1$. The exception is a «riskless» asset, say asset 0, whose (gross) rate of return, R_{0t}, is independent of the state of the world next period (i.e. certain) and determined in period $t-1$[1]. The safe return is used as the «no risk» opportunity cost of the different investments. In this respect, we are often interested in excess returns, r_{jt}, defined as the difference between asset j's uncertain return and the safe return of the riskless asset, i.e. $r_{jt} = R_{jt} - R_{0t}$. We also assume that agents have some common information set available at $t-1$ upon which they base their common judgment on the distribution of uncertain returns. For that reason, we shall work in terms of the distribution of returns conditional on the agents information set. In this context, we call $\nu_{jt} = E_{t-1}(R_{jt})$ the expected value of R_{jt} as of time $t-1$ (where $\nu_{0t} = R_{0t}$). Then the risk premium on asset j is naturally defined as $\mu_{jt} = \nu_{jt} - \nu_{0t} = E_{t-1}(r_{jt})$, i.e. the (conditional) expected value of excess returns. The risk premium measures the agents' valuation of the riskiness of the asset. Many risky assets have positive risk premia, some negative, others zero. Shares are examples of assets with positive risk premia, insurance policies of assets with negative ones, short run government bonds of assets with (almost) no risk premia. The aim of asset pricing theories is to explain why different assets have different risk premia by looking at the relationship between risk and return.

Before discussing any specific asset pricing theory, it is worth recalling how assets would be valued in an Arrow-Debreu world with a complete set of security markets for state-contingent claims. Since R_{jt} is a bundle of claims on the numeraire good contingent on the possible states of the world at time t, the asset price would simply be the sum across states of the asset payoff in each state times the price of the corresponding state-contingent claim. If we now define state prices per unit probability as the prices of the claims divided by the conditional probability of the associated states, and represent them by the random variable ζ_t, the asset price is simply the probability weighted sum across states (or expected value) of the payoffs times these redefined state prices. Since we have normalised the assets so that they cost one unit, we will have that:

$$1 = E_{t-1}(R_{jt}\,\zeta_t) \qquad [1]$$

[1] Most asset pricing theories and their tests can be easily modified to accommodate situations when a riskless asset is not available. However, we shall not discuss such versions here to avoid too lengthy an exposition.

This is basic equilibrium relationship that we mentioned in the introduction to the first part of this survey. Notice that ζ_t could also be understood as a common stochastic discount factor that discounts uncertain payoffs differently across states of the world. This interpretation is reinforced by the fact that we can rewrite the above expression for today's asset price as $1 = E_{t-1}(R_{jt})\, E_{t-1}(\zeta_t) + \mathrm{cov}_{t-1}(R_{jt}, \zeta_t)$. The first term uses $E_{t-1}(\zeta_t)$ to discount the espected payoff; the second is a risk adjustment[2]. For the riskless asset, the above expression gives $R_{0t} = 1/E_{t-1}(\zeta_t)$. Using this result we get:

$$\mu_{jt} = \mathrm{cov}_{t-1}(R_{jt}, -\zeta_t R_{0t}) \qquad [2]$$

That is, the risk premium on an asset is proportional to the covariance between that asset and the stochastic discount factor. Hence, the risk that is priced is not the total variance of the asset, but rather the part of that variance that is correlated with the common stochastic discount factor. As we shall see, this is a common theme in all asset pricing theories.

Unfortunately, the Arrow-Debreu world is too general a framework for empirical work. Besides, as we said in the first part of this review, without further restrictions on ζ_t, the basic equilibrium relation for asset returns can hardly be falsified by the data (see e.g. Hansen and Jagannathan (1991)). For take any finite set of non-redundant securities whose returns are represented by the random vector $\boldsymbol{R}_t$, with conditional mean ν_t and conditional covariance matrix Σ_t. Then, assuming that the matrix $\Sigma_t + \nu_t \nu_t$ has full rank, the portfolio $\zeta_t^* = \ell (\Sigma_t + \nu_t \nu_t)^{-1} \boldsymbol{R}_t$, with ℓ a vector of $1-s$, will trivially satisfy $E_{t-1}(R_{jt} \zeta_t^*) = 1$ and all the other expressions that we have derived from it *for all the assets in our sample.*

Hence, what all empirically orientated asset pricing theories must do at the end of the day is to propose a candidate stochastic discount factor (or implicit set of state prices), ζ_t, that satisfies the above expression and leads to testable implications. The empirically relevant question is then whether the candidate discount factor is consistent with the data.

2. Capital Asset Pricing Model

2.1. *Theory*

The CAPM was developed independently by Sharpe (1964), Lintner (1965) and Mossin (1966) in the 1960's, and it is closely related to portfolio selection theories. In particular, it can be regarded as the equilibrium version of mean-variance analysis. An intuitive way to derive it is to assume that each and every period each agent effectively decides the portfolio allocation of his current wealth by maximizing a derived utility function which

[2] ζ_t is also related to the equivalent martingale measure that allows us to write the asset pricing expression without any risk adjustment, as if agents were risk neutral.

only depends on the conditional mean and variance of next period returns on his chosen portfolio, $R_t^k = (1 - \sum_{j=1}^{N} a_{jt-1}^k) R_{0t} + \sum_{j=1}^{N} a_{jt-1}^k R_{jt}$, where a_{jt-1}^k $(j = 1, 2,...N)$ are the proportions of his wealth invested in each risky asset and $(1 - \sum_{j=1}^{N} a_{jt-1}^k)$ the proportion invested in the riskless asset. Importantly, we assume that there are no transaction costs or other impediments to trade, and in particular, that a_{jt-1}^k is totally unrestricted, so that some a_{jt-1}^k could be negative if the agent decides to short-sell (i.e. borrow) the $j - th$ asset. Then the expected return on the individual's k portfolio will be $\nu_t^k = E_{t-1}(R_t^k) = (1 - \sum_{j=1}^{N} a_{jt-1}^k) \nu_{0t} + \sum_{j=1}^{N} a_{jt-1}^k \nu_{jt}$, and its variance $\sigma_t^k = V_{t-1}(R_t^k) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{it-1}^k a_{jt-1}^k \sigma_{ijt}$, where $\sigma_{ijt} = \text{cov}_{t-1}(R_{it}, R_{jt})$.

Specifically, it is assumed that given their wealth, the agents problem can be expressed as

$$\max_{a_{jt-1}^k} \; \nu_t^k - 1/2\, \alpha_t^k \sigma_t^k \qquad [3]$$

where α_t^k is a *positive* risk aversion parameter, which indicates that they prefer a higher expected return for a given variance, but a lower variance for a given expected return.

A crucial element of the CAPM is the market portfolio, a_{Mt-1}, which costs one unit and holds the *risky* assets in amounts proportional to their *fixed* supplies. If we combine the first order conditions of the agents allocation problem [3] with market clearing, it can be proved that the $N \times 1$ vector of risk premia must satisfy the cross-sectional restriction

$$\mu_t = \alpha_t \Sigma_t a_{Mt-1} \qquad [4]$$

where α_t is a measure of aggregate risk aversion, and Σ_t the $N \times N$ matrix of variances and covariances of asset returns (i.e. $\{\Sigma_t\}_{ij} = \sigma_{ijt}$). Given that the return for the market portfolio is $R_{Mt} = \sum_{j=1}^{N} a_{Mjt-1} R_{jt}$, the above expresssion implies:

$$\mu_{Mt} = \alpha_t \sigma_{MMt} \qquad [5]$$

where $\mu_{Mt} = E_{t-1}(r_{Mt})$ and $\sigma_{MMt} = V_{t-1}(R_{Mt})$. This is the well-known risk-return trade-off: higher expected returns for the market as a whole can only be obtained if there is an increase in the risk (measured by the variance) of the market. It allows us to interpret α_t as the market price of risk, i.e. the number of units of expected value that we are willing to give away to reduce variance by one unit.

Equation [4] also implies that for each asset (or portfolio of assets):

$$\mu_{jt} = \alpha_t \text{cov}_{t-1}(R_{jt}, R_{Mt}) \qquad [6]$$

This is an equilibrium condition. It says that the risk premium on an asset is higher the higher the covariance of that asset with the market portfolio, and the higher the price of risk. Notice that is covariance, not variance which determines risk premia, in this case the covariance with the market portfolio. Importantly, the price of market risk, α_t, is the same for all assets. Comparing [6] with [1], it is clear that in the CAPM the stochastic discount factor, ζ_t, is inversely proportional to the return of the market portfolio (see also section 5 below).

This result can perhaps be better understood in terms of the following argument. Let us define β_{jMt} as the (theoretical) coefficient in the conditional regression of (the unanticipated component of) returns for the *j-th* asset, $u_{jt} = r_{jt} - \mu_{jt}$, on (the unexpected component for) the market as a whole, $u_{Mt} = r_{Mt} - \mu_{Mt}$. Thus, $\beta_{jMt} = \text{cov}_{t-1}(u_{jt}, u_{Mt})/V_{t-1}(u_{Mt})$ measures the sensitivity of the return of the *jth* asset to the market. Writing u_{jt} as:

$$u_{jt} = \beta_{jMt}\, u_{Mt} + \varepsilon_{jt} \qquad [7]$$

we are able to decompose the conditional variance of r_{jt}, σ_{jjt}, into the sum of a systematic or market-risk component, $\beta^2_{jMt}\,\sigma_{MMt}$, and a specific (or unsystematic) risk element $\omega_{jjt} = V_{t-1}(\varepsilon_{jt})$. Specific risk reflects those unexpected events affecting a given asset (e.g. shares in some firm) which are peculiar to it, or to closely related assets (e.g. shares in similar firms). By contrast, systematic risk captures those economy-wide unexpected events which threaten all assets.

Combining [5], [6] and [7], we get:

$$r_{jt} = \beta_{jMt}\, \mu_{Mt} + \beta_{jMt}\, u_{Mt} + \varepsilon_{jt} \qquad [8]$$

which says that the risk premium of any asset is its market beta times the risk premium for the market as a whole. Therefore, only the component of risk which is correlated with the market portfolio is priced in the CAPM; the variance of the specific component for the asset is not priced as it can be diversified away completely. In this respect, high beta assets are highly correlated with the market, and hence their riskiness, as measured by σ_{jjt}, is more difficult to diversify. As a consequence, their risk premia must be higher. In contrast, a negative beta produces a negative risk premium due to the fact that a security whose returns are inversely related to the market portfolio acts as an insurance policy. Besides, we can have a very risky security with risk premium that is (almost) zero if its returns are (almost) uncorrelated with those of the market portfolio, i.e. if its beta is (almost) 0. Obviously, the riskless asset has a zero beta. However, the riskiness of the market portfolio is undiversifiable. In this sense, the market portfolio has a beta of 1.

If we now recall that the risk premium on the market portfolio is proportional to its variance we have that

$$\mu_{jt} = \beta_{jMt}\, \alpha_t\, \sigma_{MMt} \qquad [9]$$

so for all assets (the absolute value of their) expected returns will be higher the higher the average degree of risk aversion and the higher the volatility of the market portfolio. Under risk neutrality, $\alpha_t = 0$, and the risk premia for all assets are zero. In this respect, it is important to mention that only one risk neutral agent is required for all risk premia to be zero[3].

Notice that since the mathematics of the mean-variance portfolio frontier implies that $\mu_{jt} = \beta_{jet}\, \mu_{et}$, where R_{et} refers to any efficient portfolio other than the riskless asset (see e.g. Huang and Litzenberger (1988)), at the end of the day, the only restriction that the CAPM actually imposes is that the market portfolio is mean-variance efficient, i.e. that given σ_{MMt}, no other combination of the risky assets produces a higher expected return.

2.2. *Empirical Tests*

Tests of the CAPM are problematic. The main critique is due to Roll (1977). The problem is that the market portfolio includes many other assets on top of tradable financial securities such as stocks and bonds; crucially, it must also include claims on real assets, human capital, etc., that are associated with some of the most important decisions taken by individuals during their lifetime, but for which little or no data is available. In this sense, the CAPM is generally untestable unless strong assumptions are made on the correlation between the observed market portfolio and the true one[4]. However, Stambaugh (1982) found that traditional tests of the CAPM were rather robust to expanding the market portfolio so as to include not only stocks, but also bonds, real estate and consumer durables. Nevertheless, some important investments, such as human capital, were still excluded.

Given that the CAPM assumes that agents choose their portfolios based on the conditional distribution of next period returns, another important problem is whether the frequency of available observations coincides with the length of investors' planning horizon.

The other main problem is how to model changes in the betas over time. Apart from tractability, some assumptions are necessary for the tests to be meaningful. For example, consider the risk-return relationship for the market as a whole, $\mu_{Mt} = \alpha_t\, \sigma_{MMt}$. If we do not restrict α_t in some sense, this expression becomes an identity. The usual assumption is that α_t is constant. In this respect, several methods have been proposed to estimate the unobservable σ_{MMt}. For example, Merton (1980) proposed using lagged squared returns as a proxy for σ_{MMt}. French, Schwert and Stambaugh (1987) constructed proxies for monthly volatility as the variance of daily

[3] If α_t^k were 0 for some agent, he would take an infinite large position in the asset with highest expected return and infinite short positions in all other assets. The only possible equilibrium would then be $\mu_{jt} = 0\ \forall j$.

[4] Most of the literature implicitly assumes that our proxy has perfect correlation with the true market portfolio. Kandel and Stambaugh (1987) and Shanken (1987) consider tests of the CAPM conditional on some other specific assumption about this correlation.

returns over the month. Then they obtained σ_{MMt} by fitting ARIMA models to this series to obtain the predictable components. Alternatively one can use the GARCH-M model of Engle, Lilien and Robins (1987). Other methods have been proposed, for example instrumental variables by Campbell and Shiller (1988) or semi-parametric by Pagan and Ullah (1988). The general finding is that the price of risk α is not very precisely determined. Unfortunately, this could simply reflect the fact that α is not constant over time.

However, the CAPM was mainly intended to price a cross-section of assets. On this basis, Bollerslev, Engle and Wooldridge (1988) used excess returns on three assets (a 6-month Tbill, a long term bond and a stock) to estimate a multivariate version of the ARCH-M model, $r_t = \alpha\Sigma_t a_{Mt-1} + u_t$, where $E_{t-1}(u_t) = 0$ and $V_{t-1}(u_t) = \Sigma_t$. They obtained sensible parameter estimates but rejected some of the restrictions. An alternative way of estimating the CAPM could be developed along the lines of Braun, Nelson and Sunier (1990). They estimate a bivariate model for the returns of the market and a portfolio, and model the different parameters as a function of the information set. Unfortunately, the estimation of models that take into account time variation in second conditional moments is hampered by the large number of parameters involved. Hence only a small number of assets can be considered without strong restrictions on Σ_t (as in section 3).

But if we are only interested in testing the CAPM restrictions without estimating the model, the latent variable approach of Gibbons and Ferson (1985) offers a possible way out. Assume for simplicity that β_{jMt} is time-invariant and that the risk premium for the market μ_{Mt} is a linear function of some variables, z_{t-1}, in the information set. From equation [8] we can derive the testable implication that the coefficients in the regression of r_{jt} on z_{t-1} should be proportional across equations. In fact, we can easily drop the assumption of linearity if we notice that the CAPM implies that for any reference asset i, $\mu_{jt} = (\beta_{jM}/\beta_{iM})\,\mu_{it}$ for all j. Given a set of variables in the information set (i.e. instruments), z_{t-1}, we can write this formally as $E\,[(r_{jt} - \gamma_{ji}\, r_{it}) \mid z_{t-1}] = 0$, where $\gamma_{ji} = \beta_{jM}/\beta_{iM}$. Hansen's (1982) overidentifying restrictions test can then be used to check the validity of the CAPM without specifying how μ_{Mt} depends on the information set (see Ferson, Foerster and Keim (1993)).

The traditional way of estimating and testing the CAPM, however, is based on the unconditional distribution of returns. However, such an approach poses substantial interpretation difficulties, as conditional versions of the CAPM do not generally imply clear cut restrictions on the unconditional moments (see e.g. Hansen and Richard (1987)). To avoid these problems, for the remaining of this section we shall assume that asset returns have constant conditional variances and covariances. The lack of realism of this assumption is partly compensated in the empirical studies by considering only short sample periods.

The initial empirical tests were based on the idea that in the CAPM the relative pricing of different assets is only determined by their betas. If we knew the betas, test statistics could be straightforwardly calculated on the basis of panel data-type regressions. Unfortunately, this is not the case. In practice, the tests are carried out using a two-pass regression. In the first pass, betas are estimated from a time-series regression of returns on the market portfolio. The rationale for such a procedure comes from [8], which can be re-written as the empirical market model

$$r_{jt} = \beta_{jM}\, r_{Mt} + \varepsilon_{jt} \qquad [10]$$

where $\text{cov}_{t-1}(r_{Mt}, \varepsilon_{jt}) = 0$. This equation also allows us to obtain a measure of unsystematic risk from the OLS residual variance. In the second step, returns are cross-sectionally regressed on the estimated betas to obtain expected returns.

Black, Jensen and Scholes (1972) used a modified version of this procedure in which they cross-sectionally regressed the time-series average of the asset returns on the estimated betas. Their results were apparently impressive, because nearly 100% of the cross sectional variability in average returns could be explained by the variability in the betas. However, there are several problems with their procedure. The main one is that using estimates of betas as regressors induces what is essentially a generated regressor version of the error in variables problem. It is particularly serious if N is large relative to the sample size, and does not go away as far as inference is concerned even if the number of observations goes to infinity while the number of assets remains fixed (see Shanken (1992a)). One way to reduce it is to use portfolios rather than individual assets, preferably consisting of assets with similar betas. Actually, Black, Jensen and Scholes (1972) used beta-ranked portfolios, but since they used estimated betas to form portfolios, they overestimated the betas of portfolios with high betas and underestimated the betas of portfolios with low betas.

The Fama and MacBeth (1973) solution is to rank the assets according to betas computed using an initial sample period, and recompute the betas using another non-overlapping period's data. Then they carried out the cross-sectional regression using a third sample period. In particular, they estimated

$$r_{pt} = a_t + b_t\, \hat{\beta}_p + c_t\, \hat{\beta}_p^2 + d_t\, s_p^2 + v_{pt} \qquad [11]$$

where s_p^2 is the average of the OLS residual variance obtained from [10] for all assets in portfolio p. According to the CAPM, $a_t = c_t = d_t = 0$ and $b_t = \mu_{Mt}$. In fact, they used a five year rolling procedure to recompute $\hat{\beta}_p$ and s_p^2. The hypothesis that they tested was whether the time-series average of the a_t–s, c_t–s and d_t–s was zero. They accepted the last two but find some evidence that the average a_t was positive. However, they interpreted this rejection as evidence in favour of a version of the CAPM in which lending at the risk-free rate is allowed, but borrowing is not.

From the econometric point of view, their estimates still suffer from a errors in variables problem, although presumably smaller. More importantly, the ε_{jt}−s in [11] are contemporaneously correlated and heteroskedastic, so that a GLS procedure would be more appropriate, not only in terms of efficiency, but mainly because OLS gives the wrong standard errors. Besides, that c_t or d_t is zero on average is a very weak test of the model.

These tests may also have low power for a different reason. Neither Black, Jensen and Scholes (1972) nor Fama and Macbeth (1973) directly tested the basic CAPM restriction that the market portfolio is mean-variance efficient. Such a direct test been recently proposed by Gibbons, Ross and Shanken (1989) on the basis of equation [10]. If the CAPM is correct, this expression should not include any intercept[5]. Hence, an obvious test of the efficiency of the proxy for the market portfolio used can be obtained by including a constant and seeing if it is zero for all assets. Gibbons, Ross and Shanken (1989) and Shanken (1985) convincingly argue that if the distribution of the idiosyncratic risk components conditional on r_{Mt} is multivariate normal with constant variance, the standard F test for multivariate regression should be used. The reason is that such a test is more reliable than the Wald, Likelihood Ratio or Lagragange Multiplier versions because it has an exact finite sample distribution, both under the null and under the alternative. This point is relevant even for large sample sizes if the number of assets is close to the number of observations, since in that case the asymptotic χ^2 distribution of the other tests is not very reliable as a finite sample approximation. Unfortunately, whether or not the distributional assumptions are correct is important because in some cases the F distribution may no longer be reliable, even asymptotically. If so, a more sensible approach would be to use a Wald test with a robust estimate of the covariance matrix (see MacKinlay and Richardson (1991)).

Using the same data set as Black, Jensen and Scholes (1972), Gibbons, Ross and Shanken could not reject that an equally-weighted portfolio was ex-ante mean-variance efficient. However, their results were sensitive to the period and index composition used. Besides, univariate t statistics indicated that the significance of the zero-intercept restriction changed substantially from some portfolios to others. In this respect, they found that high-beta portfolios earned too little on average to be consistent with the theory, and low beta portfolios too much.

Many other studies have found that other variables which in principle should have no role in the CAPM are useful in explaining excess returns. Examples are size, leverage, dividend yields, earnings-price ratios, book-to-market ratios, or month of the year. This is true even after controlling for beta (see Fama and French (1992, 1993) and the references therein). As usual, these anomalies could be interpreted as a failure of market rationa-

[5] Note that the regression intercept corresponds to Jensen's alpha as defined in performance measurement.

lity or of the asset pricing model used. In this respect, there has been considerable effort to try and explain these anomalies in terms of other asset pricing models. The APT is one such theory.

3. Arbitrage Pricing Theory

3.1. Theory

The starting point of Ross' (1976) APT is the assumption that the systematic risk component of returns for the countable collection of existing assets can be decomposed into the sum of the influences of some k economy wide sources of uncertainty. More explicitly, it is assumed that for all $N = 1, 2, \ldots$, the vector of asset returns, R_{Nt}, is generated according to the following conditional factor model of risk:

$$R_{Nt} = v_{Nt} + u_{Nt} = v_{Nt} + B_{Nt} f_t + \varepsilon_{Nt} \qquad [12a]$$

where f_t are the k common risk influences or factors and ε_{Nt} the idiosyncratic risk components. Expanded for any asset, R_{jt} [12a] becomes:

$$R_{jt} = v_{jt} + \beta_{j1t} f_{1t} + \ldots + \beta_{jkt} f_{kt} + \varepsilon_{jt} \qquad [12b]$$

The (full column rank) matrix B_{Nt} contains the factor loadings or betas, which measure the sensitivity of the assets to the different sources of systematic risk (i.e. $\{B_{Nt}\}_{ji} = \beta_{jit}$. An important point to notice is that different assets will usually have differing sensitivities to the common sources of risk.

The APT is generally vague about what the factors are; as far as the theory is concerned, the only requirement is that they are pervasive, i.e. they should affect most asset returns. In principle, they could be directly related to observable macroeconomic shocks such as innovations in industrial production or unanticipated inflation, or they could be left unspecified, in which case they simply capture the undiversifiable cross-sectional correlation of returns.

For convenience and without loss of generality, it is usually assumed that the factors represent conditionally orthogonal influences, so that we can understand β_{jit} as the conditional univariate regression coefficient $\mathrm{cov}_{t-1}(u_{jt}, f_{it}) / V_{t-1}(f_{it})$. It is also often assumed that the idiosyncratic terms for any two assets are uncorrelated, although this condition is stronger than required. The specific factors can be mildly correlated if a law of large numbers still applies cross-sectionally (see Chamberlain and Rothschild (1983)). Again, by construction the idiosyncratic or asset-specific noises, ε_{Nt}, are uncorrelated with the factors. We shall call Λ_t the diagonal covariance matrix of the factors, and Ω_{Nt} the covariance matrix of the idiosyncratic terms.

In order to derive the asset pricing restrictions implied by the APT in an intuitive manner, let us define arbitrage portfolios as those which involve zero net wealth. That is, if we call a_{Nj} the amount of asset j in such a portfolio, then $\sum_{j=1}^{N} a_{Nj} = a_N' l_N = 0$, where l_N is a vector of N ones. Let us now construct (a sequence of) arbitrage portfolios that do not contain any systematic risk by choosing their weights so that $a_N' B_{Nt} = 0$. If we also choose these arbitrage portfolios to be well diversified ($a_N' a_N \to 0$ so that $a_{jN}^2 = o\,(1/N)$), then, by the law of large numbers $V_{t-1}(a_N' \varepsilon_{Nt}) = a_N' \Omega_{Nt} a_N$ will tend to zero as the number of assets goes to infinity. The reason is that $a_N' \varepsilon_{Nt}$ is a (weighted) average of the ε_{jt}-s which are assumed to be only mildly correlated. As a consequence, the gross returns for such portfolios will be $a_N' R_{Nt} = a_N' \nu_{Nt} + a_N' B_{Nt} f_t + a_N' \varepsilon_{Nt} \approx a_N' \nu_{Nt}$ since they contain no systematic risk, and (almost) no unsystematic risk either. These portfolios are then (almost) safe assets since their returns have (almost) no risk at all. But since these arbitrage «riskless» portfolios cost nothing, an arbitrage argument implies that in equilibrium their returns must also be (almost) zero, i.e. $a_N' \nu_{Nt} \approx 0$. Otherwise it would be possible to make enormous amounts of money: for instance, if $a_N' \nu_{Nt}$ were significantly greater than zero, we could obtain positive safe returns by holding this asset at zero cost.

The above derivation is valid for any well diversified (sequence of) portfolios satisfying $a_N' [l_N \mid B_{Nt}] = 0$. But then, the condition $a_N' \nu_{Nt} \approx 0$ implies that the vector of expected returns, ν_{Nt}, must be (almost) a linear combination of the columns of the matrix $[l_N \mid B_{Nt}]$. That is:

$$\nu_{Nt} \approx l_N \pi_{0t} + B_{Nt} \pi_t \qquad [13a]$$

or

$$\nu_{jt} \approx \pi_{0t} + \pi_{1t} \beta_{j1t} + \ldots + \pi_{kt} \beta_{jkt} \qquad [13b]$$

Strictly speaking, the APT only provides an approximate pricing theory. In this respect, it can be proved that the (cross-sectional) average of the (squared) pricing errors resulting from the difference between the left and right hand sides of [13] is negligible. This does not mean, though, that all pricing errors will be small, which clearly complicates inference procedures (see Reisman (1992) or Shanken (1982, 1992)). To avoid such problems, we shall assume in what follows that an equilibrium version of the APT holds so that the pricing errors dissapear. A necessary and sufficient condition for exact APT pricing is that the implicit stochastic discount factor, ζ_t, is well diversified (see e.g. Chamberlain (1983)).

Given that for the riskless asset, R_{0t}, $\beta_{0it} = 0$ for $i = 1,.,k$, the exact version of [13] implies that $\nu_{0t} = \pi_{0t}$, which finally allows us to write the cross-equation pricing restriction implied by the equilibrium version of the APT:

$$\mu_{Nt} = \nu_{Nt} - 1_N \nu_{0t} = B_{Nt} \pi_t \qquad [14a]$$

or

$$\mu_{jt} = \pi_{1t} \beta_{j1t} + \ldots + \pi_{kt} \beta_{jkt} \qquad [14b]$$

That is, the risk premium for any asset is a linear combination of its betas with weights that are common to all assets.

One way to interpret π_t is to consider k sequences of portfolios, $f^R_{Nt} = A'_{Nt} \, r_{Nt}$, whose weights tend to satisfy $A'_{Nt} \, B_{Nt} = I_k$ as the number of assets, N, increases. These are factor mimicking portfolios because *in the limit* they will have a unit loading on one factor and zero loadings on the others. Then, it is not difficult to see that $E_{t-1} \, (f^R_{Nt})$ converges to π_t, so that π_{it} can be understood as the risk premium of the *i-th* (limiting) factor representing portfolio. From this viewpoint, the APT says that the risk premium on an asset is a linear combination of the risk premia associated with the factors, with weights given by the sensitivities of that asset to the factors.

Combining equation and we can write the model for returns as:

$$r_{Nt} = B_{Nt} \, (f_t + \pi_t) + \varepsilon_{Nt} = B_{Nt} f^R_t + \varepsilon_{Nt} \qquad [15]$$

where f^R_t are the returns on the (limiting) factor representing portfolios.

It is also possible to write for the APT an expression analogous to [6] for the CAPM. From the definition of β_{jit}, we can re-write [14b] as:

$$\mu_{jt} = \tau_{1t} \, cov_{t-1} \, (r_{jt}, f_{1t}) + \ldots + \tau_{kt} \, cov_{t-1} \, (r_{jt}, f_{kt}) \qquad [16]$$

where $\tau_{it} = \pi_{it} \, / \, \lambda_{it}$, $i = 1,.,k$. Given that π_{it} can be understood as the risk premia associated with factor i, and λ_{it} is the corresponding variance, we can interpret τ_{it} as the «price of risk» for this factor, i.e. the amount of expected return that agents are willing to give away to reduce its variability by one unit. Equation [16] then says that risk premia is a linear combination of the covariances of asset returns with the factors weighted by the corresponding «prices of risk». Importantly, the risk prices depend on the factors, not on the assets. Again, what is priced in the APT is covariance, not variance. In this case, the relevant measure is covariance with the (orthogonal) systematic risk components, which are undiversifiable. Once more, idiosyncratic variance, ω_{jjt}, is not priced.

Notice also the similarity of [15] with equation [10] for the traditional CAPM. As a matter of fact, if $k = 1$ and the factor representing portfolio were actually the market portfolio [15], would reduce to [10]. However, the CAPM is not necessarily a one factor model. As a matter of fact, the equilibrium version of the APT is also compatible with the CAPM under more general circumstances. To see why, let us assume that the market portfolio is well-diversified. This assumption implies that effectively, the different asset returns are correlated with the market only through the common factors. Specifically,

$$cov_{t-1} \, (r_{jt}, r_{Mt}) = \beta_{j1t} \, \beta_{M1t} \, \lambda_{1t} + \ldots + \beta_{jkt} \, \beta_{Mkt} \, \lambda_{kt}$$

According to the CAPM, the risk premium for an asset is $\alpha_t \, cov_{t-1} \, (r_{jt}, r_{Mt})$, while according to the equilibrium version of the APT, $\mu_{jt} = \beta_{j1t} \, \tau_{1t} \, \lambda_{1t} + \ldots + \beta_{jkt} \, \tau_{kt} \, \lambda_{kt}$. Hence, both expressions are equal if and only if the prices of risk for each factor satisfy the restriction $\tau_{it} = \alpha_t \, \beta_{Mit}$. Intuitively, this restriction says that in the CAPM, the factors are only priced through their influence on the market portfolio. It is important to emphasize, though, that the CAPM pricing restrictions are robust to the crucial assumption in the APT that returns follow a conditional factor structure.

3.2. Empirical tests

Since in the above derivation the factors did not have to be specified, the APT is in principle very general. In this respect, an advantage of the APT over the CAPM is that it is not necessary to have data on all available assets to estimate it, because the market portfolio does not play any significant role. On the other hand, we must have the right number of factors, and if these are prespecified, they must be correctly so.

For a fixed number of assets, tests for the number of factors can be easily obtained given that the exact factor structure imposes overidentifying restrictions on the covariance matrix of returns (see e.g. Roll and Ross (1980)). However, it seems that more factors are required as the number of assets increases (see e.g. Dhrymes, Friend and Gultekin (1984)). This finding is confirmed by Luedecke (1984) using an alternative testing procedure. However, the results of both papers could be due to the fact that the reliability of the asymptotic distribution of such tests is poor for large N relative to T. This is an open question, and no consensus view on the number of factors exists as yet.

Once more, time variation is the betas is an important practical problem, but again, without any restrictions on their variability, the APT is untestable. In this respect, Engle, Ng and Rothschild (1990) and King, Sentana and Wadhwani (1993) have recently estimated conditional versions of the APT under the assumption that for an appropriate choice of (conditionally orthogonal) factors, betas and prices of risk are constant over time. The former look at holding returns for eight short term US T-bill (between 2 and 12 months) together with aggregate returns for the New York stock market; the latter at stock returns for sixteen national stock markets around the world. Notice that the assumption on the prices of risk implies that the risk premia associated with each factor is a proportional to its variance, λ_{it}, with a constant of proportionality that does not change over time. Equation [14] can then be written as:

$$\mu_{Nt} = B_N \Lambda_t \tau \qquad [17a]$$

or

$$\mu_{jt} = \tau_1 \lambda_{1t} \beta_{j1} + \ldots + \tau_k \lambda_{kt} \beta_{jk} \qquad [17b]$$

Both studies model the conditional variances of the common factors according to GARCH processes. The main difference is in the factor specification. Engle, Ng and Rothschild (1990) pre-specify two factors: one is an equally-weighted portfolio of bond returns, the other the stock market. These authors find that idiosyncratic variance is not priced, as indicated by the theory. Nevertheless, they find that lagged values of the spread between short and long term bonds help predict returns. King, Sentana and

Wadhwani (1993) considered both observable factors related to the innovations in macroeconomic variables, and unobservable factors, internal to the returns process. In addition to testing whether idiosyncratic variance is priced, they also test the APT cross-equation rectriction that the price of risk coefficients τ should be the same for all countries. In both instances, they reject. One way of interpreting their results is as evidence against international stock market integration. They also find that the observable factors only have a small, but significant, explanatory role.

As in the CAPM, the traditional way of testing the APT is based on the unconditional distribution of returns. Since this may be once more rather problematic, we shall assume again that the conditional covariance matrix of returns is constant over time. Then, if we choose $V_{t-1}(f_t) = I_k$ without loss of generality, [17] reduces to:

$$\mu_{Nt} = B_N \tau$$

or

$$\mu_{jt} = \tau_1 \beta_{j1} + \ldots + \tau_k \beta_{jk}$$

Roll and Ross (1980) estimated B using factor analytic techniques. Having obtained these estimates, for every period, they cross-sectionally regressed r_{jt} on $\hat{\beta}_{j1},\ldots,\hat{\beta}_{jk}$ to obtain the coefficients $\tilde{\tau}_{1t},\ldots,\tilde{\tau}_{kt}$. In this respect, a GLS estimator was employed to take into account the (cross-sectional) correlation and heteroskedasticity in the residuals. Then they tested whether the (time-series) average of $\tilde{\tau}_{it}$ is 0. To reduce the errors in variables problem, they used portfolios, rather than individual assets. They also tested whether idiosyncratic variance is priced. Overall, their results were encouraging.

If we knew the (limiting) factor representing portfolios, equation [15] will provide an obvious alternative way of testing the APT by checking that the regression intercept is indeed zero. In general, though, f_t^R is not directly observed, only some finite sample counterparts. Lehmann and Modest (1988) carry out such regression tests using several alternative factor representing portfolios computed from a large cross-section of assets. They find that like the CAPM, the APT cannot explain the size effect. However, they claim that it can account for the dividend yield anomaly. Connor and Korajczyk (1988) perform very similar tests using asymptotic principal components as factor mimicking portfolios. These factor representing portfolios are particularly easy to compute when N is much larger than T. Their results suggest that the APT does a better job than the CAPM of explaining the January seasonality in excess returns, but a worse job of explaining the size effect. More recently, Mei (1993) has proposed an alternative semiautoregressive approach to estimate the factors. His resuls confirm that the APT describes returns better than the CAPM, but that there is still some misspricing left.

Chen, Roll and Ross (1986) took a different route and fully specified the factors. In particular, they used four factors given by the innovations in the spread between a long-term yield and a short interest rate, the rate of inflation, the spread between yields on low grade corporate bonds and treasury bills and the growth in industrial production. Their results were also encouraging.

Most other CAPM tests that we discussed in section 2.3. can be easily adapted to the APT. For instance, the Gibbons, Ross and Shanken (1989) procedure can also be applied if we use as regressors some factor representing portfolios, f^R_{Nt}, and test the joint significance of the intercepts. More recently, Ferson and Harvey (1991) have estimated an APT-type model by two-pass regression procedures. In the first step, they compute the betas by rolling regressions of asset returns on observed economic variables. Then, they do cross-sectional regressions to estimate the prices of risk for the following observation. They investigate whether the predictability found in stock returns under the assumption of constant expected returns can be explained by changes in betas and risk premia for the factors. Their results suggest that the latter accounts for a significant proportion of the predictability.

In general, the APT seems to do a better job than the CAPM in explaining the differences in risk premia across-assets. Unfortunately, it is not a full success. Besides, the APT vagueness about what the factors are is a somewhat undesirable property, since as economists, we would like to relate stock returns more closely to the real economy.

4. Consumption Capital Asset Pricing Model

4.1. *Theory*

From an economist's point of view, the Concumption CAPM is the most natural asset pricing theory. It is perhaps surprising that it was the last one to be derived (see Breeden (1979), Lucas (1978) or Rubinstein (1976)). The starting point is an individual who cares about consumption not only today but also in the future, and who chooses today's consumption level, c_{t-1}, and the proportions of his savings invested in each asset, a^k_{jt-1} ($j = 1, 2, \ldots N$) by maximizing his lifetime expected utility function U_{t-1} $(c^k_{t-1}, c^k_t, \ldots, c^k_{t+1}, \ldots) = E_{t-1}$ $[V_{t-1}, (c^k_{t-1}, c^k_t, \ldots, c^k_{t+1}, \ldots)]$ subject to an intertemporal budgetary restriction. Specifically, agent *k's* programme is:

$$\max_{c^k_{t-1},\, a^k_{jt-1}} E_{t-1} \left[V_{t-1}, (c^k_{t-1}, c^k_t, \ldots, c^k_{t+1}, \ldots)\right]$$

subject to the intertemporal budget constraint

$$w^k_t = (w^k_{t-1} - c^k_{t-1})\, R^k_t = (w^k_{t-1} - c^k_{t-1}) \left[\left(1 - \sum_{j=1}^{N} a^k_{jt-1}\right) R_{0t} + \sum_{j=1}^{N} a^k_{jt-1} R_{jt}\right]$$

where w^k_{t-1} is his wealth at the end of period $t - 1$ when decisions are taken, $(w^k_{t-1} - c^k_{t-1})$ his net savings, and R^k_t the overall return on his investment.

The main reason for investing in a risky asset now and deferring today's consumption is to transfer part of that purchasing power to the future and consume more tomorrow. In this context, we can consider the following variational argument. At the optimum, if we save a unit of consumption today, invest it in the *j-th* asset and consume the asset payoffs tomorrow, it must be the case that the marginal utility of foregone consumption must be equal to the expected gain from consuming the proceeds next period. Under the assumption of additive intertemporal separability, we must have that:

$$\partial V_{t-1}/\partial c_{t-1} = E_{t-1}\,(R_{jt}\,\partial V_{t-1}/\partial c_t) \qquad [18]$$

This is the so-called stochastic Euler equation. Defining $S_t = dc_{t-1}/dc_t = (\partial V_{t-1}/\partial c_t) \,/\, (\partial V_{t-1}/\partial c_{t-1})$ as the marginal rate of intertemporal substitution in consumption between $t-1$ and t, the Euler equation implies:

$$E_{t-1}\,(R_{jt}\,S_t)= 1 \qquad [19]$$

which says that the expected value of the product of the marginal rate of substitution in consumption and the gross return must be equal to 1 for every asset and for every time period[6].

S_t can then be identified as the stochastic discount factor, ζ_t. The intuition is very simple if we go back to the Arrow-Debreu world. Standard micro theory says that the equilibrium price of a contingent claim in terms of the numeraire good should be the common marginal rate of substitution between such claim and the numeraire.

Once more, we get:

$$\mu_{jt} = -R_{0t}\,cov_{t-1}\,(R_{jt}, S_t) \qquad [20]$$

That is, the risk premium on an asset is proportional to the covariance between that asset and the marginal rate of substitution in consumption. This is an equilibrium condition similar to [6] for the CAPM and [16] for the APT. Under the implicit assumption that $\partial V_t/\partial c_t$ declines with consumption, S_t will be lower (higher) than expected if c_t is higher (lower) than expected. Hence if asset j tends to do well when times are good (i.e. when consumption is high so S_t is low), then S_t and R_{jt} are negatively correlated. The risk premium will have to be positive to convince an individual to hold an asset like that. On the other hand, if an asset pays handsomely in bad times (i.e. when consumption is low so S_t is high), S_t and R_{jt} are positively correlated, and the risk premium will be negative. The idea is that such an asset acts like an insurance policy for bad times and agents are willing to pay for it. Although perhaps a bit surprising at first sight, this derives from the fact that the only risk which is priced is that related to covariances, not variances. For example, if there were a very volatile asset whose payoffs were nevertheless (almost) independent of whether times are

[6] By the law of iterated expectations, this relationship is also true unconditionally, i.e. $E\,(R_{jt}\,S_t) = 1$, and hence it is independent of each consumer's information in this unconditional sense.

bad or good, i.e. $cov_{t-1}(R_{jt}, S_t) \simeq 0$, then $\mu_{jt} \cong 0$, despite the fact that this asset returns would show great variability. The intuition is that (almost) all its risk would be specific and could be diversified away completely.

4.2. *Empirical tests*

The best known tests of the CCAPM were performed by Hansen and Singleton (1983). They assumed that the representative agent's utility function was of the Constant Relative Risk Aversion (or isoelastic) type:

$$V_{t-1}(c^k_{t-1}, c^k_t, \dots, c^k_{t+i}, \dots) = \begin{cases} \sum_{j=0}^{\infty} \rho^{-j} (1-\varphi)^{-1} c^{1-\varphi}_{t-1+j} & \text{for } \varphi \neq 1 \\ \sum_{j=0}^{\infty} \rho^{-j} \log c_{t-1+j} & \text{for } \varphi = 1 \end{cases} \qquad [21]$$

Such a functional form is not only analytically very convenient, but also its parameters ρ and φ can be given a direct economic interpretation. ρ is the rate of time-preference or discount factor: a large ρ indicates that consumers value consumption today much more than tomorrow. φ plays a double role: φ is the coefficient of relative risk aversion, φ^{-1} the intertemporal elasticity of substitution in consumption between period $t-1$ and t (i.e. $\varphi^{-1} = \partial E_{t-1}(\Delta \log c_t) / \partial \log R_{0t}$). If φ is 0, agents are risk neutral, i.e. they are indifferent at risk. Otherwise, for positive φ, the larger this parameter is, the more they dislike taking on risks. On the other hand, a small φ^{-1} means that consumers prefer to have a smooth consumption pattern during their lives; a large one, that they don't mind large changes in life-style from one period to the next.

For this particular utility function, we can write the Euler equation as

$$E_{t-1}[R_{jt}(c_t/c_{t-1})^{-\varphi}] = \rho \qquad [22]$$

Assuming that c_t/c_{t-1} and R_{jt} are jointly log-normally distributed and homoskedastic (given I_{t-1}), $\log R_{jt} - \varphi \Delta \log c_t$ will be normally distributed with mean $E_{t-1}(\log R_{jt}) - \varphi E_{t-1}(\Delta \log c_t)$, and variance $\varphi^2\sigma_{cc} + \sigma_{jj} - 2\varphi\sigma_{jc}$, where σ_{cc}, σ_{jj} are the (time-invariant) variances of consumption growth and $\log R_{jt}$ respectively, and σ_{jc} their covariance. The Euler equation then implies that $E_{t-1}(\log R_{jt}) - \varphi E_{t-1}(\Delta \log c_t) + 1/2(\varphi^2\sigma_{cc} + \sigma_{jj} - 2\varphi\sigma_{jc}) = \log\rho$. As a consequence,

$$E_{t-1}(\log R_{jt}) = [\varphi E_{t-1}(\Delta \log c_t) + \log\rho - 0.5\varphi^2\sigma_{cc}] + [\varphi\sigma_{jc} - 0.5\sigma_{jj}] \qquad [23]$$

Notice that the first three terms of this equation are the same for all assets. In fact, they are equal to the (log) safe interest rate:

$$\log R_{0t} = \varphi E_{t-1}(\Delta \log c_t) + \log\rho - 0.5\varphi^2\sigma_{cc} \qquad [24]$$

as $\sigma_{00} = \sigma_{0c} = 0$. This is the inverse relationship to the intertemporal consumption models which make $\Delta \log c_t$ a function of the interest rate. It says that the risk free rate is higher the smaller the degree of intertemporal substitution (i.e. the smaller φ^{-1} is), and the more agents prefer consumption today relative to tomorrow's (i.e. the higher ρ is). Besides, if expected growth in consumption is high so that tomorrow should bring good times, then the incentive to save has to be higher.

But all this tells us nothing about risk premia, only about savings and intertemporal substitution. If we subtract [24] from [23] we get:

$$E_{t-1}\,(\log R_{jt}) - \log R_{0t} = \varphi\sigma_{jc} - 0.5\sigma_{jj} \qquad [25]$$

which is the same as [4] except for the term $-\ 0.5\sigma_{jj}$ arising from working in logs, rather than in levels. As [20] before, [25] simply says that risk premia are higher the higher the covariance between consumption growth and returns, and the higher the degree of risk aversion. In the special case of risk neutrality, $\varphi = 0$, risk premia are zero and all expected returns are equal to the discount factor ρ.

In order to estimate [23] by maximum likelihood, Hansen and Singleton (1983) specified $E_{t-1}\,(\Delta\log c_t)$ as a function of variables known at $t-1$. Their results essentially rejected the cross-asset restrictions on the coefficient of $\Delta\log c_t$, although the point estimates were reasonably sensible.

Hansen and Singleton (1982) used a different approach. Instead of assuming any distribution for consumption and asset returns, they exploited the moment restrictions implicit in the Euler equation:

$$E_{t-1}\,[\rho^{-1}\,(c_t/c_{t-1})^{-\varphi}\,R_{jt} - 1] = 0 \qquad [26]$$

by means of Hansen's (1982) Generalized Method of Moments estimator. Again, they rejected the overidentifying restrictions.

The intuitive reason for the rejection (see Mehra and Prescott (1985)) is that while aggregate consumption is very smooth, asset returns are highly volatile and their covariances with consumption growth then to be small. Of course, the rejection could also be due to misspecification of the utility function, and in particular, to the inadequacy of the assumption of additive intertemporal separability. Recently, there has been some work using more general utility functions that drop such an assumption (see e.g. the habit formation model of Constantinides (1990), or the non-expected recursive utility framework of Epstein and Zin (1989, 1991) discussed in section 5), but the more recent empirical results do not seem to fully explain the data. Alternatively, the rejections may well be due to data problems (see e.g. Breeden, Gibbons and Litzenberger (1989)). For instance, consumption is measured with error, it is time-aggregated and/or time-deformed, consumption of asset market participants may be poorly proxied by aggregate consumption if some agents are liquidity constraint, etc. For this reasons there have been attempts to substitute consumption data and use only data on returns, i.e. to internalise the Consumption CAPM (see Campbell (1993)).

Recently, an alternative, nonparametric approach to testing the CCAPM (or indeed any other asset pricing model) has been proposed by Hansen and Jagannathan (1991). On the basis of asset market data and without reference to any specification of preferences, they construct mean-variance bounds for potential stochastic discount factors, ζ_t, in the following way. We saw in section 1 that $\zeta_t^* = \ell' (\Sigma_t + \nu_t \nu_t')^{-1} R_t$ will correctly price all the assets in R_t by construction. That is, $E_{t-1} (\zeta_t^* R_t) = 1$. But since the «true» stochastic discount factor ζ_t also prices the assets correctly by assumption, it must be the case that $E_{t-1} [(\zeta_t - \zeta_t^*) R_t] = 0$. This means that ζ_t^* is the fitted value from the regression of ζ_t on R_t. If R_t includes a conditionally safe asset, we will have from least squares theory that $E_{t-1} (\zeta_t) = E_{t-1} (\zeta_t^*)$ and $V_{t-1} (\zeta_t) \geqslant V_{t-1} (\zeta_t^*)$. Hence, the set of first two moments for ζ_t compatible with the data is a half-line in mean-variance space starting at $1/R_{0t}$, $V_{t-1} (\zeta_t^*)$. If R_{0t} is not observed, we can still draw the boundary region for many potential values of R_{0t} (see Hansen and Jagannathan (1991) for details).

The adequateness of alternative parametrized versions of the CCAPM (or indeed any other empirically orientated asset pricing model) can then be judged by checking whether the first two (conditional) moments of their implied ζ_t—s lie within the bounds. The results in Gallant, Hansen and Tauchen (1990), Hansen and Jagannathan (1991) and Cochrane and Hansen (1992) cast some doubts on the class of preferences generally considered in the literature.

5. Nesting the CAPM and CCAPM

Although the traditional and consumption versions of the CAPM are apparently very different, they coincide under the assumption of logarithmic risk preferences. The intuition is that since consumption is a constant fraction of wealth in this case, $S_t = c_{t-1}/\rho c_t$ is equal to the reciprocal of $w_t^k/(w_{t-1}^k - c_{t-1}^k) = R_t^k$, the return on the portfolio of the representative individual, i.e. the «market» portfolio. This confirms that in the CAPM the stochastic discount factor is inversely related to the market return. Straightforward manipulations then imply that:

$$E_{t-1} (\log R_{jt}) - \log R_{0t} = -0.5\sigma_{jj} + \sigma_{jM} \qquad [27]$$

so that the market price of risk is one.

Actually, it is possible to nest both the CAPM and the Consumption CAPM by dropping the assumption of expected utility along the lines of Epstein and Zin (1989, 1991). They separate risk aversion and intertemporal substitution by assuming that the representative agent's intertemporal allocation problem can be reduced to the riskless selection between today's consumption and the certainty equivalent of future consumption utility. While risk preferences regulate how certainty equivalents are obtained, intertemporal substitution parameters determine dynamic choice.

They define the representative agent's objective function recursively as:

$$U_{t-1} = \{(1 - \tfrac{1}{\rho})\, c_{t-1}^{1-1/\gamma} + \tfrac{1}{\rho}\, [E_{t-1} (U_t^{1-\varphi})]^{\frac{1-1/\gamma}{1-\varphi}}\}^{\frac{1}{1-1/\gamma}}$$

where is again the rate of time-preference or discount factor, and φ the coefficient of relative risk aversion. In contrast, γ now denotes the intertemporal elasticity of substitution in consumption between period $t-1$ and t, which is no longer necessarily equal to φ^{-1}. However, if $\varphi = \gamma^{-1}$, we go back to the time-separable power utility function in [21].

If we define $\theta = (1 - \varphi)/(1 - 1/\gamma)$, the Euler equation is in this case:

$$E_{t-1} \{[\rho^{-1} (c_t/c_{t-1})^{-1/\gamma}]^{\theta} R_{Mt}^{\theta-1} R_{jt}\} = 1$$

which for the market portfolio reduces to:

$$E_{t-1} \{[\rho^{-1} (c_t/c_{t-1})^{-1/\gamma} R_{Mt}]^{\theta}\} = 1$$

where θ can be either positive or negative. Assuming that (given I_{t-1}) c_t, R_{Mt} and R_{jt} are log-normally distributed and homoskedastic, a similar argument to that in Hansen and Singleton (1983) leads to:

$$E_{t-1} (\log R_{jt}) = \theta \log\rho + \theta/\gamma\, E_{t-1} (\Delta \log c_t) + (1 - \theta)\, E_{t-1} (\log R_{Mt}) -$$
$$0.5[(\theta/\gamma)^2 \sigma_{cc} + (\theta - 1)^2 \sigma_{MM} + \sigma_{jj} - 2\theta/\gamma\, (\theta - 1)\, \sigma_{MC} - 2\theta/\gamma \sigma_{jc} + 2\, (\theta - 1)\, \sigma_{jM}]$$

Subtracting the corresponding equation for the riskless asset we finally obtain:

$$E_{t-1} (\log R_{jt}) - \log R_{0t} = -0.5\, \sigma_{jj} + \theta/\gamma\, \sigma_{jc} + (1 - \theta)\, \sigma_{jm}$$

This expression is a generalization of both [25] and [27]. Again, if the utility function is logarithmic, then $\theta = 0$, and we obtain the static CAPM with constant risk premia in [27]. In general, the covariance with both consumption and the market are important in determining risk premia.

Although the empirical results obtained with this model clearly indicate its superiority over the expected utility framework, unfortunately, they do not show unequivocal success in explaining risk premia.

6. Conclusions

In this paper, we have reviewed the theory and empirical work related to three of the most common empirically orientated asset pricing theories currently in use: the consumption and traditional versions of the Capital Asset Pricing Model, and the Arbitrage Pricing Theory. Our goal has been to stress the similarities rather than the differences of these theories. The unifying theme is that all asset prices should be equal to the conditional expected value of the product of their uncertain payoffs and a common stochastic discount factor. In the CCAPM, the stochastic discount factor is the intertemporal marginal rate of substitution in consumption for the representative agent; in the CAPM, the reciprocal of the return on the market portfolio; in the exact version of the APT, a well diversified portfolio containing only factor risk. From the above fundamental valuation rule, it follows that the risk premium on an asset is proportional to its undiversifiable risk component, and hence, that diversifiable risk should not be priced. Undiversifiable risk is measured by the covariance of the asset at hand with economy wide risks. Again different asset pricing theories have different proxies for those common risks.

The empirical success of these theories is mixed, but undoubtedly higher for those theories that explain expected returns in terms of other assets expected returns. By and large, attempts to explain returns in terms of «macroeconomic» variables have not been very successful. Nevertheless, there are still several asset pricing anomalies to be explained.

In the late eighties it was hoped that one could explain these anomalies by using conditioning information to allow for time variation in risk and expected return. Unfortunately, empirical attempts to link time-varying volatility and changing risk premia have been somewhat disappointing so far. However, there is yet a lot to learn about the empirical properties of stock prices, especially at the level of individual firms, and the relationship between their cross-sectional structure and time series behaviour. Not surprisingly, the most recent empirical research is moving along these directions.

There is also a substantial amount of theoretical research currently going on in relaxing some of the maintained assumptions of the standard theories. Heterogeneously informed agents, incomplete markets or market frictions, are some of the directions in which theoretical asset pricing is moving. The empirical testing of such new theories will not lag much behind.

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Resumen

Este trabajo analiza tres modelos de valoración de activos con una orientación empírica: el CAPM tradicional y el de consumo, y el APT. Los tres implican que los precios de los activos son el valor esperado de sus pagos multiplicados por una tasa estocástica común de descuento. De este modo, la prima de riesgo de un activo sería proporcional a su covarianza con riesgos comunes a la economía y al riesgo diversificable no sería retribuido.

Las teorías son normalmente contrastadas comprobando si las rentabilidades descontadas apropiadamente son cero de media para varios activos simultáneamente. Empíricamente, el éxito es mayor para aquellas que explican las rentabilidades en términos de otros activos en vez de usando variables macroeconómicas. Sin embargo, existen diversas anomalías por explicar.

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